

Solutions to Math 21a Exam 2 Practice Problems - Spring 2000

Fall 1999 Exam 2

1) a) The tangent plane is the plane through the given point which is orthogonal to the gradient.

$$\text{Here, } \nabla g = (-3y + z + 2, 2y - 3x, x - 1).$$

$$\text{At } (-1, 0, -1), \text{ one has } \nabla g = (1, 3, -2).$$

Thus, the tangent plane consists of the points (x, y, z) which obey
 $x + 3y - 2z - 1 = 0$.

b) A normal vector to the plane where $x = y$ is the vector $(1, -1, 0)$. This is proportional to ∇g when $x = 1$ and $z = y + 1$.

Furthermore, a point $(1, y, z = y + 1)$ has $g = 0$ only when $y^2 - 3y + 2 = 0$, which has $y = 1$ or $y = 2$. Thus, $P = (1, 1, 2)$ or $P = (1, 2, 3)$. However, the second point must be rejected since the tangent plane at the point P is given by the equation $x = y$. So $P = (1, 1, 2)$.

c) $L(x, y, z) = x + 3y - 2z - 1$ has the same value as g at $(-1, 0, -1)$ and the same gradient.

d) Any vector \mathbf{u} which obeys $\nabla g \cdot \mathbf{u} = 0$.

Thus,

$$\mathbf{u} = (-3a + 2b, a, b) / (10a^2 + 5b^2 - 12ab)^{1/2}$$

where both a and b are not zero. For

$$\text{instance, } \mathbf{u} = \frac{(-3, 1, 0)}{\sqrt{10}}.$$

2) a) The hottest point on the surface is

$$\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right). \text{ Temp} = 20 \text{ degrees.}$$

b) The hottest point inside or on the surface is

$$\left(\frac{\sqrt{3}}{10}, 0, \frac{1}{10}\right). \text{ This is inside the moon and the temp} = 52 \text{ degrees.}$$

c) The coldest point is $\left(-\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right)$.

Temp = -20 degrees. The extreme points on the surface are obtained by solving the Lagrange multiplier equations for those points where $x^2 + y^2 + z^2 = 1$ and $\nabla T = \lambda \nabla g$ where $g(x, y, z) = x^2 + y^2 + z^2$.

$$\text{Here, } \nabla T = 10(\sqrt{3}, 0, 1) - 100(x, y, z) \text{ and } \nabla g = (2x, 2y, 2z).$$

The extreme point inside is obtained by solving for the points where $x^2 + y^2 + z^2 < 1$ and $\nabla T = \mathbf{0}$.

3) a) $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}''(0) = 2\mathbf{j}$.

b) $\mathbf{s}'(0) = -\mathbf{j}$ and $\mathbf{s}''(0) = -\mathbf{i}$.

c) Write $\nabla T = a\mathbf{i} + b\mathbf{j}$. Then we are told that

$$\nabla T \cdot \mathbf{r}'(0) = a + b = 3 \text{ and}$$

$$\nabla T \cdot \mathbf{s}'(0) = -b = -1. \text{ Thus, } b = 1 \text{ and } a = 2$$

$$\text{and } \nabla T = 2\mathbf{i} + \mathbf{j} \text{ at } (1, 0).$$

4) a) The critical points occur where

$$\nabla f = (2xy - 4y, x^2 - 4x + y^2) = \mathbf{0}.$$

These are $(2, \pm 2)$, $(0, 0)$ and $(4, 0)$.

b) $(2, 2)$ is a local minimum, $(2, -2)$ is a local maximum, $(0, 0)$ is a saddle, $(4, 0)$ is a saddle. The 2nd derivative test establishes these assertions since the matrix f'' of 2nd

derivatives has $\det(f'') > 0$ and

$\text{trace}(f'') > 0$ at $(2, 2)$, and $\det(f'') > 0$

and $\text{trace}(f'') < 0$ at $(2, -2)$, and

$\det(f'') < 0$ at $(0, 0)$ and $(4, 0)$.

c) The direction of maximum increase is

$$\frac{(1, 1)}{\sqrt{2}}.$$

That of maximum decrease is $\frac{(-1, -1)}{\sqrt{2}}$.

d) Any vector of the form $(a, 0)$ with $a \neq 0$.

- 5) The best linear approximation to f at $(0, 25)$ is
 $L(x, y) = 5 + 5x + \frac{1}{10}(y - 25)$.
 Using L to estimate f gives $5 + .5 + .03 = 5.53$.
- 6) After doing the y integration, one is left with
 integrating $\frac{1}{2}x^3(1 - x^2)$ between 0 and 1.
 The integral is $\frac{1}{24}$.
- 7) Do the y integral first. (There is no closed
 form expression for the x integral if that one is
 done first.) The range for the y integral is from
 $y = 0$ to $y = x$. The resulting x integral is for
 the function $2xe^{x^2}$ with the range going from
 $x = 0$ to $x = 1$. Changing variables to $u = x^2$
 shows that this is the same as the integral of e^u
 from $u = 0$ to $u = 1$, which is $e - 1$.

Additional Problems

8. a) $L(x, y, z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1)$
 (or $L(x, y, z) = 2x + 2y + 2z - 3$).
 b) $L(x, y, z) = y + z$.
9. 12 (Note: Since $y_r = 0$ at $r = 1$ and
 $s = -1$, you don't need to compute any y
 derivatives when using the Chain Rule.)
10. $\pi + 1$. ($w' = 4t \tan^{-1}(t) + 1$).
11. $\nabla f = (3, 2, -4)$.
12. 2
13. $\mathbf{u} = \frac{(1, 1, 1)}{\sqrt{3}}$
14. 0.
15. $2x + 2y + z - 4 = 0$.
16. $x - y + 2z - 1 = 0$.
17. $2\sqrt{2}$.

18. Absolute max: 1 at $(0, 0)$;
 absolute min: -5 at $(1, 2)$.
19. Absolute max: 4 at $(2, 0)$;
 Absolute minima along left edge ($x = 0$)
 where value is constant = 0.
 (There are also two relative minima:
 value = $\frac{3}{\sqrt{2}}$ at $(3, \pi/4)$ and $(3, -\pi/4)$.)
20. Local min at $(1, -2)$; saddle point at $(-1, -2)$.
21. Length in the x -direction: $4\sqrt{2}$;
 length in the y -direction: $3\sqrt{2}$.
22. $(3/2, 2, 5/2)$.
23. $(0, 0, 2)$ and $(0, 0, -2)$,
24. $(\pm 4/3, -4/3, -4/3)$.
25.
$$\begin{bmatrix} -30x + \frac{2 - 2x^2}{(x^2 + 1)^2} & 1 \\ 1 & 0 \end{bmatrix}$$
26. $L(x, y, z) = x + y - z - 1$.
27. Width = $(c^2 V/(ab))^{1/3}$;
 depth = $(b^2 V/(ac))^{1/3}$;
 height = $(a^2 V/(bc))^{1/3}$.
28. A normal vector to the surface is the gradient
 of $xz^2 - yz + \cos(xy) - 1$ which, at $(0, 0, 1)$, is
 the vector $\mathbf{v} = (1, -1, 0)$. Meanwhile,
 $\mathbf{r}(1) = (0, 0, 1)$ and $\mathbf{r}'(1) = (1, 1, 1)$ which has
 zero dot product with the normal vector \mathbf{v} .
 Thus, $\mathbf{r}'(1)$ lies in the tangent plane to the
 surface.

29. Set $z(x, y) = x^3 + y^3 - 9xy + 27$ as a function on \mathbf{R}^2 . Then, maxima and minima can only occur at points where ∇z is zero. Since $\nabla z = (3x^2 - 9y, 3y^2 - 9x, 0)$, this occurs where $x^2 = 3y$ and $y^2 = 3x$. Thus, at $(x, y) = (0, 0)$ and $(x, y) = (3, 3)$. The second derivative test for z'' at the point $(0, 0)$ finds z'' to have only the off-diagonal entries non-zero. (These equal 9). Thus, $\det(z''(0, 0)) < 0$ so $(0, 0)$ is a saddle. At $(3, 3)$, $\det(z'') = 241 > 0$ and $\text{trace}(z'') = 36$, so $(3, 3)$ is a minimum.

30. 1.

31. $(3 \ln 2)/2$.

32. 2.

33. $(e - 1)/2$.

34. 16.

35. 12.

36. $(1 - \ln 2) \pi$.

37. 12π .

38. $4/3 + 5\pi/8$.