Practice Problems for Exam 2 - Math 21a - Spring 2000

Fall 1999 Exam 2

1. Consider the surface where the function $g(x, y, z) = y^2 - 3xy + zx + 2x - z$ is zero.
   a) Find the tangent plane to this surface at (-1, 0, -1).
   b) Find a point P on this surface where the tangent plane is given by the equation $x = y$. Explain your reasoning.
   c) Find the best linear approximation to this function $g(x, y, z)$ at the point (-1, 0, -1).
   d) Find a direction (a unit vector) in which the directional derivative of g at (-1, 0, -1) is zero.

2. Suppose that the moon is modeled by the ball where $x^2 + y^2 + z^2 \leq 1$. If the temperature of a point (x, y, z) in or on the moon at a particular time is given by the function

$$T(x, y, z) = 50(1 - x^2 - y^2 - z^2) + 10(\sqrt{3}\,x + z)$$

   then find:
   a) The (x, y, z) coordinates of the hottest of the points on the surface of the moon.
   b) The (x, y, z) coordinates of the points in or on the moon where temperature is greatest.
   c) The (x, y, z) coordinates of the points in or on the moon where the temperature is least.
   In all cases, make sure you justify your conclusions.

3. The positions of a rat and a snail on the xy-plane are given, respectively by

$$\mathbf{r}(t) = (1 + t)\,\mathbf{i} + (t^2 + t)\mathbf{j} \quad \text{and}$$
$$\mathbf{s}(t) = \cos(t)\,\mathbf{i} + (t^3 - t)\,\mathbf{j},$$

   where t is time measured in seconds.
   a) Give the velocity and acceleration vectors for the rat at $t = 0$.
   b) Give the velocity and acceleration vectors for the snail at $t = 0$.
   c) The temperature of any point (x, y) on the plane is position dependent and thus given by a function, $T(x, y)$, measured in degrees. In particular, note that the temperature at any given point doesn't change with time. However, as the rat and snail are moving, they feel the temperature change. For example, at $t = 0$ the rat feels the temperature increase at the rate of 3 degrees per second, while the snail feels the temperature decrease at the rate of 1 degree per second. With the preceding understood, compute ∇T at the point (1, 0).

4. Let $f(x, y) = x^2 y - 4xy + y^3/3$.
   a) Find all critical points of f.
   b) Identify the points found in Part a as local maxima, minima or saddle points. Justify your answers.
   c) Find the directions of maximum increase and decrease of f at (1, -3).
   d) Find the tangent vector to the level curve of f through (2, 4).

5. Use the best linear approximation to $f(x, y) = e^x y^{1/2}$ at a convenient point to estimate the value of f at (0.1, 25.3).

6. Integrate the function $f(x, y) = x^3 y$ over the region where $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$.

7. Calculate the integral $\iint_R 2e^{x^2}\,dA$ where R is the triangle where $0 \leq y \leq 1$ and $y \leq x \leq 1$. Thus, the vertices of R are (0, 0), (1, 0) and (1, 1) in the x-y plane.

Additional Problems

8. Find the linear approximation to the function $f(x, y, z) = xy + yz + zx$ at
   a) the point (1, 1, 1).
   b) the point (1, 0, 0).

9. Find $\frac{\partial w}{\partial r}$ at $(r, s) = (1, -1)$ if
   $w = (x + y + z)^2 \cos(\pi (x + 4y)/3)$
   and (x, y, z) are the following functions of r and s:
   $x = r - s$, $y = \cos(r + s)$, and $z = \sin(r + s)$.

10. Find $\frac{dw}{dt}$ at $t = 1$ for $w = 2ye^x - \ln z$, and for $x = \ln(t^2 + 1)$, $y = \tan^{-1} t$ and $z = e^t$.

11. Find ∇f at (1, 1, 1) for
    $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x$.

12. Find the directional derivative at (0, 0, 0) in the direction of $\mathbf{u} = (2, 1, -2)/3$ for the function $f(x, y, z) = 3e^x \cos(yz)$.

13. Find the unit vector giving the direction of most rapid increase at the point (1, 1, 1) for the function
    $f(x, y, z) = \ln(xy) + \ln(yz) + \ln(xz)$.

14. Estimate the amount by which the function $g(x, y, z) = x + x\cos(z) - y\sin(z) + y$
    will change if one moves from (2, -1, 0) a distance 0.2 units towards the point (0, 1, 2).

15. Find the equation for the tangent plane at (0, 1, 2) to the surface where (x, y, z) obeys the equation $\cos(\pi x) - x^2 y + e^{xz} + yz - 4 = 0$.

16. Find the equation for the tangent plane at (1, 2, 1) to the surface where $z^2 = y - x$.

17. The derivative of a certain function $f(x, y, z)$ at a certain point P is greatest in the direction of the vector (1, 1, -1) and in this direction, the derivative is $2\sqrt{3}$. What is the derivative at P of this same function in the direction of the vector (1, 1, 0)?

18. Find the maximum and minimum values of $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the triangle in the first quadrant boundaries lie where
    $x = 0$, $y = 2$ and $y = 2x$.

19. Find the maximum and minimum values of $f(x, y) = (4x - x^2)\cos(y)$ where (x, y) obey the conditions $0 \le x \le 3$ and $-\pi/4 \le y \le \pi/4$.

20. Find the maxima, minima and saddle point of a function $f(x, y)$ (if any), given that the partial derivatives of f are $f_x = 9x^2 - 9$
    and $f_y = 2y + 4$.

21. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse where $x^2/16 + y^2/9 = 1$ whose sides are parallel to the coordinate axis.

22. Find the point on the plane $x + 2y + 3z = 13$ which is closest to the point (1, 1, 1).

23. Find the points on the surface $z^2 = xy + 4$ closest to the origin.

24. The surface of a space probe is in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$. Meanwhile, the temperature on the surface of this probe is given by
    $T(x, y, z) = 8x^2 + 4yz - 16z + 600$.
    Find the hottest point on the surface.

25. Find the 2×2 matrix f'' of 2nd order partial derivatives of
    $f(x, y) = x + xy - 5x^3 + \ln(x^2 + 1)$.

26. Find the best linear approximation to $f(x, y, z) = xy + 2yz - 3xz$ at (1, 1, 0).

27. A closed, rectangular box is to have volume V cm^3. The cost of the material used in the box is a cents/cm^2 for the top and bottom, b cents/cm^2 for the front and back, and c cents/cm^2 for the remaining sides.
In terms of V, find (a, b, c) which minimize the cost of the box.

28. Show that the curve $\mathbf{r}(t) = (\ln t, t \ln t, t)$ is tangent to the surface $xz^2 - yz + \cos(xy) = 1$ at the point (0, 0, 1).

29. Show that the only possible maxima and minima of z on the surface $z = x^3 + y^3 - 9xy + 27$ occur at (0, 0) and (3, 3). Show that (0, 0) is neither a maximum nor a minimum.
Is (3, 3) a maximum or minimum?

30. Integrate $f(x, y) = x + y + 1$ over the region where $-1 \le x \le 1$ and $-1 \le y \le 0$.

31. Integrate $f(x, y) = \dfrac{x}{y}$ over the region in the first quadrant cut out by the four lines whose equations are $y = x$, $y = 2x$, $x = 1$, and $x = 2$.

32. Integrate $f(x, y) = \dfrac{\sin y}{y}$ over the region where $0 \le x \le \pi$ and $x \le y \le \pi$.
(Hint: Be careful when you choose the order in which to do this integral.)

33. Integrate $f(x, y) = x^2 e^{xy}$ over the region where $0 \le y \le 1$ and $y \le x \le 1$.

34. Find the volume of the solid where $0 \le x \le 3$, $0 \le y \le 2$, and $0 \le z \le 4 - y^2$.

35. Find the area of the region where $0 \le y \le 6$ and $y^2/3 \le x \le 2y$.

36. Find the integral of $2/(1 + (x^2 + y^2)^{1/2})$ on the region where $-1 \le x \le 0$ and $-(1 - x^2)^{1/2} \le y \le 0$.
(Hint: Use polar coordinates.)

37. Find the area enclosed by 1 leaf of the rose $r = 12 \cos 3\theta$.

38. The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid, right cylinder whose height is given by $z = r \cos\theta$.
Find the cylinder's volume.