into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \Rightarrow Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft^3/s, and the remaining 1065 ft^3/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

11 Review

CONCEPT CHECK

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.
   (b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy-plane.
2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.
3. $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.
4. (a) See Definition 11.2.3.
   (b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.
5. (a) See (2) and (3) in Section 11.3.
   (b) See the discussion preceding Example 2 on page 769.
   (c) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.
6. See the statement of Clairaut's Theorem on page 773.
7. (a) See (2) in Section 11.4.
   (b) See (19) and the preceding discussion in Section 11.6.
   (c) See the discussion following Example 6 on page 787.
8. See (3) and (4) and the accompanying discussion in Section 11.4. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b). Thus it is the linear function which best approximates f near (a, b).
9. (a) See Definition 11.4.7.
   (b) Use Theorem 11.4.8.
10. See (10) and the associated discussion in Section 11.4.

11. See (2) and (3) in Section 11.5.

12. See (7) and the preceding discussion in Section 11.5.

13. (a) See Definition 11.6.2. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P. (See Figure 5 in Section 11.6.)

(b) See Theorem 11.6.3.

14. (a) See (8) and (13) in Section 11.6.

(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

(c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

15. (a) f has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).

(b) f has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) By Theorem 11.7.2, if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. See (3) in Section 11.7.

18. (a) See Figure 11 and the accompanying discussion in Section 11.7.

(b) See Theorem 11.7.8.

(c) See the procedure outlined in (9) in Section 11.7.

19. See the discussion beginning on page 822; see the discussion preceding Example 5 on page 826.

▲ TRUE–FALSE QUIZ ▲

1. True. $f_y(a, b) = \lim_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ from Equation 11.3.3. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$.

4. True. From Equation 11.6.14 we get $D_{\mathbf{k}} f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 11.2.3.

6. False. See Exercise 11.4.40(a).

7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 11.7.2, $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle = \langle 0, 0\rangle = \mathbf{0}$.

8. False. The limit does not exist because the function is not defined on the line $y = x$, and so we have a path approaching the point $(1, 1)$ along which f does not approach $\frac{1}{2}$.

9. False. $\nabla f(x, y) = \langle 0, 1/y\rangle$.

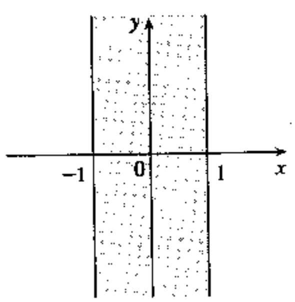
10. True. This is part (c) of the Second Derivatives Test (11.7.3).

11. True. $\nabla f = \langle \cos x, \cos y\rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos\theta| \le 1$, so $|\nabla f| \le \sqrt{2}$. Now $D_{\mathbf{u}} f\,(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}| \cos\theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}} f(x, y)| \le \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.

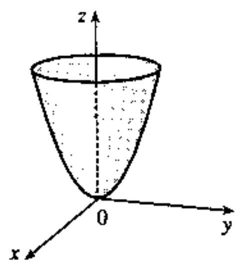
12. False. See Exercise 11.7.29.

EXERCISES

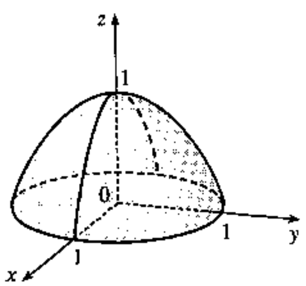
1. The domain of $\sin^{-1} x$ is $-1 \le x \le 1$ while the domain of $\tan^{-1} y$ is all real numbers, so the domain of $f(x, y) = \sin^{-1} x + \tan^{-1} y$ is $\{(x, y) \mid -1 \le x \le 1\}$.



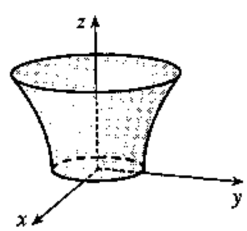
2. $D = \{(x, y, z) \mid z \ge x^2 + y^2\}$, the points on and above the paraboloid $z = x^2 + y^2$.



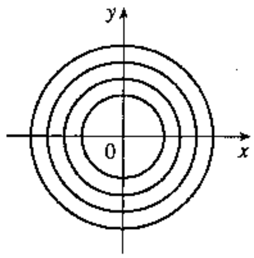
3. $z = f(x, y) = 1 - x^2 - y^2$, a paraboloid with vertex $(0, 0, 1)$.



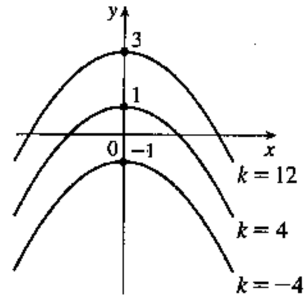
4. $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$, so $z \ge 0$ and $1 = x^2 + y^2 - z^2$. Thus the graph is the upper half of a hyperboloid of one sheet.



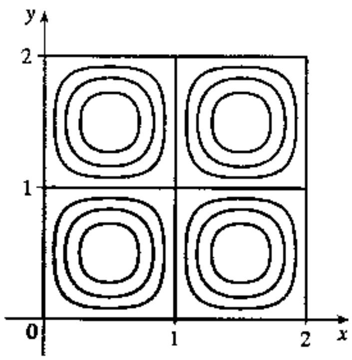
5. Let $k = e^{-c} = e^{-(x^2+y^2)}$ be the level curves. Then $-\ln k = c = x^2 + y^2$, so we have a family of concentric circles.



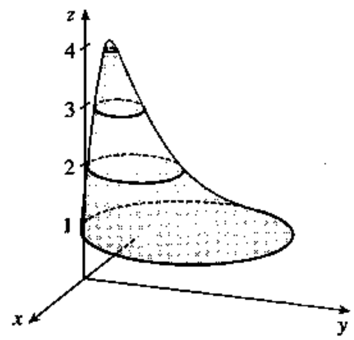
6. $k = x^2 + 4y$ or $4(y - k/4) = -x^2$, a family of parabolas with vertex at $(0, k/4)$.



7.



8.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1,1)$, we use direct substitution to evaluate the limit: $\lim\limits_{(x,y)\to(1,1)} \dfrac{2xy}{x^2+2y^2} = \dfrac{2(1)(1)}{1^2+2(1)^2} = \dfrac{2}{3}$.

10. As $(x,y) \to (0,0)$ along the x-axis, $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ along this line. But $f(x,x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x,y) \to (0,0)$ along the line $x = y$, $f(x,y) \to \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6,4) = \lim\limits_{h\to 0} \dfrac{T(6+h,4) - T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6,4) \approx \dfrac{T(8,4) - T(6,4)}{2} = \dfrac{86-80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4) - T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5\,^\circ\mathrm{C}/\mathrm{m}$. Similarly, $T_y(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h) - T(6,4)}{h}$, which we can approximate with $h = \pm 2$: $T_y(6,4) \approx \dfrac{T(6,6) - T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$,

$T_y(6,4) \approx \dfrac{T(6,2) - T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0\,^\circ\mathrm{C}/\mathrm{m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 11.6.9, $D_\mathbf{u}T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_\mathbf{u}T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35\,^\circ\text{C}/\text{m}$.

Alternatively, we can use Definition 11.6.2: $D_\mathbf{u}T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_\mathbf{u}T(6,4) \approx \dfrac{T(8,6) - T(6,4)}{2\sqrt{2}} = \dfrac{80 - 80}{2\sqrt{2}} = 0$,

$D_\mathbf{u}T(6,4) \approx \dfrac{T(4,2) - T(6,4)}{-2\sqrt{2}} = \dfrac{74 - 80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have

$D_\mathbf{u}T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1\,^\circ\text{C}/\text{m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[T_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h) - T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h) - T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6) - T(6,6)}{2} = \dfrac{80 - 75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6) - T(6,6)}{-2} = \dfrac{68 - 75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \dfrac{T(8,2) - T_x(6,2)}{2} = \dfrac{90 - 87}{2} = 1.5$, $T_x(6,2) \approx \dfrac{T(4,2) - T(6,2)}{-2} = \dfrac{74 - 87}{-2} = 6.5$.

Averaging these values, we estimate $T_{xx}(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \dfrac{T_x(6,6) - T_x(6,4)}{2} = \dfrac{3.0 - 3.5}{2} = -0.25$,

$T_{xy}(6,4) \approx \dfrac{T_{xx}(6,2) - T_x(6,4)}{-2} = \dfrac{4.0 - 3.5}{-2} = -0.25$. Averaging these values, we have

$T_{xy}(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$\begin{aligned} T(x,y) &\approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) \\ &= 3.5x - 3y + 71 \end{aligned}$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1\,^\circ\text{C}$.

13. $f(x,y) = \sqrt{2x + y^2} \quad\Rightarrow\quad f_x = \frac{1}{2}\left(2x + y^2\right)^{-1/2}(2) = \dfrac{1}{\sqrt{2x + y^2}}$,

$f_y = \frac{1}{2}\left(2x + y^2\right)^{-1/2}(2y) = \dfrac{y}{\sqrt{2x + y^2}}$

14. $u = e^{-r}\sin 2\theta \quad\Rightarrow\quad u_r = -e^{-r}\sin 2\theta$, $u_\theta = 2e^{-r}\cos 2\theta$

15. $g(u,v) = u\tan^{-1} v \quad\Rightarrow\quad g_u = \tan^{-1} v$, $g_v = \dfrac{u}{1 + v^2}$

16. $w = \dfrac{x}{y-z} \quad \Rightarrow \quad w_x = \dfrac{1}{y-z}$, $w_y = x(-1)(y-z)^{-2} = -\dfrac{x}{(y-z)^2}$,
$w_z = x(-1)(y-z)^{-2}(-1) = \dfrac{x}{(y-z)^2}$

17. $T(p,q,r) = p\ln(q+e^r) \quad \Rightarrow \quad T_p = \ln(q+e^r)$, $T_q = \dfrac{p}{q+e^r}$, $T_r = \dfrac{pe^r}{q+e^r}$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S-35) + 0.016D \quad \Rightarrow$
$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35-35) \approx 3.587$, thus in $10\,^\circ$C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y) = 4x^3 - xy^2 \quad \Rightarrow \quad f_x = 12x^2 - y^2$, $f_y = -2xy$, $f_{xx} = 24x$, $f_{yy} = -2x$, and $f_{xy} = f_{yx} = -2y$.

20. $z = xe^{-2y} \quad \Rightarrow \quad z_x = e^{-2y}$, $z_y = -2xe^{-2y}$, $z_{xx} = 0$, $z_{yy} = 4xe^{-2y}$, and $z_{xy} = z_{yx} = -2e^{-2y}$.

21. $f(x,y,z) = x^k y^l z^m \quad \Rightarrow \quad f_x = kx^{k-1}y^l z^m$, $f_y = lx^k y^{l-1} z^m$, $f_z = mx^k y^l z^{m-1}$,
$f_{xx} = k(k-1)x^{k-2}y^l z^m$, $f_{yy} = l(l-1)x^k y^{l-2} z^m$, $f_{zz} = m(m-1)x^k y^l z^{m-2}$, $f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m$,
$f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1}$, and $f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}$.

22. $v = r\cos(s+2t) \quad \Rightarrow \quad v_r = \cos(s+2t)$, $v_s = -r\sin(s+2t)$, $v_t = -2r\sin(s+2t)$, $v_{rr} = 0$,
$v_{ss} = -r\cos(s+2t)$, $v_{tt} = -4r\cos(s+2t)$, $v_{rs} = v_{sr} = -\sin(s+2t)$, $v_{rt} = v_{tr} = -2\sin(s+2t)$, and
$v_{st} = v_{ts} = -2r\cos(s+2t)$.

23. $u = x^y \quad \Rightarrow \quad u_x = yx^{y-1}$, $u_y = x^y \ln x$ and $(x/y)u_x + (\ln x)^{-1} u_y = x^y + x^y = 2u$.

24. $\rho = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad \rho_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $\rho_{xx} = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$.

By symmetry, $\rho_{yy} = \dfrac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ and $\rho_{zz} = \dfrac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$. Thus

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = 2\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}} = \frac{2}{\rho}.$$

25. (a) $z_x = 6x + 2 \quad \Rightarrow \quad z_x(1,-2) = 8$ and $z_y = -2y \quad \Rightarrow \quad z_y(1,-2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x-1) + 4(y+2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1,-2,1)$ is $\langle 8, 4, -1 \rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are
$\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$.

26. (a) $z_x = e^x \cos y \quad \Rightarrow \quad z_x(0,0) = 1$ and $z_y = -e^x \sin y \quad \Rightarrow \quad z_y(0,0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x-0) + 0(y-0)$ or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0,0,1)$ is $\langle 1, 0, -1 \rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1 - t$, and symmetric equations are $x = 1 - z$, $y = 0$.

27. (a) Let $F(x,y,z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2,-1,1) = 4$, $F_y(2,-1,1) = -4$, $F_z(2,-1,1) = -6$. From Equation 11.6.19, an equation of the tangent plane is $4(x-2) - 4(y+1) - 6(z-1) = 0$ or equivalently $2x - 2y - 3z = 3$.

(b) From Equation 11.6.20, symmetric equations for the normal line are $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$.

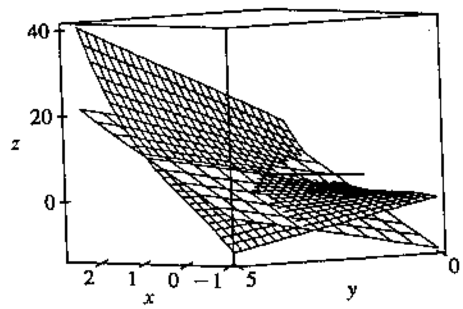
28. (a) Let $F(x,y,z) = xy + yz + zx$. Then $F_x = y+z$, $F_y = x+z$, $F_z = x+y$, so $F_x(1,1,1) = F_y(1,1,1) = F_z(1,1,1) = 2$. From Equation 11.6.19, an equation of the tangent plane is $2(x-1) + 2(y-1) + 2(z-1) = 0$ or equivalently $x + y + z = 3$.

(b) From Equations 11.6.20, symmetric equations for the normal line are $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$ or equivalently $x = y = z$.

29. (a) $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + u^2\,\mathbf{j} + v^2\,\mathbf{k}$ and the point $(3,4,1)$ corresponds to $u = 2$, $v = 1$. Then $\mathbf{r}_u = \mathbf{i} + 2u\,\mathbf{j} \Rightarrow \mathbf{r}_u(2,1) = \mathbf{i} + 4\,\mathbf{j}$ and $\mathbf{r}_v = \mathbf{i} + 2v\,\mathbf{k} \Rightarrow \mathbf{r}_v(2,1) = \mathbf{i} + 2\,\mathbf{j}$. A normal vector to the surface at $(3,4,1)$ is $\mathbf{r}_u \times \mathbf{r}_v = 8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so an equation of the tangent plane there is $8(x-3) - 2(y-4) - 4(z-1) = 0$ or equivalently $4x - y - 2z = 6$.

(b) A direction vector for the normal line through $(3,4,1)$ is $8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so a vector equation is $\mathbf{r}(t) = (3\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}) + t\,(8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k})$, and the corresponding parametric equations are $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$.

30. Let $f(x,y) = x^3 + 2xy$. Then $f_x(x,y) = 3x^2 + 2y$ and $f_y(x,y) = 2x$, so $f_x(1,2) = 7$, $f_y(1,2) = 2$ and an equation of the tangent plane is $z - 5 = 7(x-1) + 2(y-2)$ or $7x + 2y - z = 6$. The normal line is given by $\dfrac{x-1}{7} = \dfrac{y-2}{2} = \dfrac{z-5}{-1}$ or $x = 7t+1$, $y = 2t+2$, $z = -t+5$.



31. $F(x,y,z) = x^2 + y^2 + z^2$, $\nabla F(x_0,y_0,z_0) = \langle 2x_0, 2y_0, 2z_0\rangle = k\langle 2,1,-3\rangle$ or $x_0 = k$, $y_0 = \frac{1}{2}k$ and $z_0 = -\frac{3}{2}k$. But $x_0^2 + y_0^2 + z_0^2 = 1$, so $\frac{7}{2}k^2 = 1$ and $k = \pm\sqrt{\frac{2}{7}}$. Hence there are two such points: $\left(\pm\sqrt{\frac{2}{7}}, \pm\frac{1}{\sqrt{14}}, \mp\frac{3}{\sqrt{14}}\right)$.

32. $z = x^2 \tan^{-1} y \Rightarrow dz = \left(2x \tan^{-1} y\right) dx + \left[x^2/(y^2+1)\right] dy$

33. $f(x,y,z) = x^3\sqrt{y^2+z^2} \Rightarrow f_x(x,y,z) = 3x^2\sqrt{y^2+z^2}$, $f_y(x,y,z) = \dfrac{yx^3}{\sqrt{y^2+z^2}}$, and $f_z(x,y,z) = \dfrac{zx^3}{\sqrt{y^2+z^2}}$, so $f(2,3,4) = 8(5) = 40$, $f_x(2,3,4) = 3(4)\sqrt{25} = 60$, $f_y(2,3,4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2,3,4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z) &\approx f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \tfrac{24}{5}(y-3) + \tfrac{32}{5}(z-4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then

$$\begin{aligned} (1.98)^3\sqrt{(3.01)^2 + (3.97)^2} &= f(1.98, 3.01, 3.97) \approx 60(1.98) + \tfrac{24}{5}(3.01) + \tfrac{32}{5}(3.97) - 120 \\ &= 38.656 \end{aligned}$$

34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017\text{ m}^2$ or 170 cm^2.

(b) $z = \sqrt{x^2+y^2}$, $dz = \dfrac{x}{\sqrt{x^2+y^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $\dfrac{dw}{dt} = \dfrac{1}{2\sqrt{x}}\left(2e^{2t}\right) + \dfrac{2y}{z}\left(3t^2+4\right) + \dfrac{-y^2}{z^2}\,(2t) = e^t + \dfrac{2y}{z}\left(3t^2+4\right) - 2t\,\dfrac{y^2}{z^2}$

36. $\dfrac{\partial z}{\partial u} = (-y\sin xy - y\sin x)(2u) + (-x\sin xy + \cos x) = \cos x - 2uy\sin x - (\sin xy)(x+2uy)$,

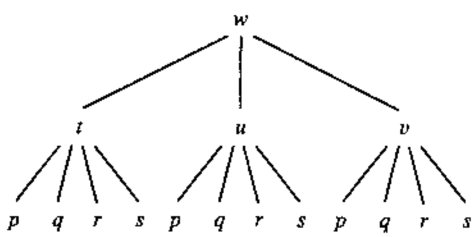
$\dfrac{\partial z}{\partial v} = (-y\sin xy - y\sin x)(1) + (-x\sin xy + \cos x)(-2v) = -2v\cos x + (\sin xy)(2vx - y) - y\sin x$

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108.$

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $\dfrac{\partial z}{\partial x} = 2xf'\left(x^2-y^2\right)$, $\dfrac{\partial z}{\partial y} = 1 - 2yf'\left(x^2-y^2\right)$ $\left[\text{where } f' = \dfrac{df}{d\left(x^2-y^2\right)}\right]$. Then

$y\,\dfrac{\partial z}{\partial x} + x\,\dfrac{\partial z}{\partial y} = 2xyf'\left(x^2-y^2\right) + x - 2xyf'\left(x^2-y^2\right) = x.$

40. $A = \frac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\,\dfrac{dx}{dt} + (x\sin\theta)\,\dfrac{dy}{dt} + (xy\cos\theta)\,\dfrac{d\theta}{dt}\right]$. So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$,

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8\text{ in}^2\text{/s}.$

41. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\, y + \dfrac{\partial z}{\partial v}\,\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}y + \frac{\partial^2 z}{\partial v\partial u}\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\partial v}y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\partial v}x\right)\\ &= x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Thus

$$\begin{aligned}x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\ &= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\frac{\partial^2 z}{\partial u\partial v} = 2v\frac{\partial z}{\partial v} - 4uv\frac{\partial^2 z}{\partial u\partial v}\end{aligned}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

42. $F(x,y,z) = e^{xyz} - yz^4 - x^2z^3 = 0$, so $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yze^{xyz} - 2xz^3}{xye^{xyz} - 4yz^3 - 3x^2z^2} = \dfrac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$

and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xze^{xyz} - z^4}{xye^{xyz} - 4yz^3 - 3x^2z^2} = \dfrac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$.

43. $\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}}, \dfrac{xz^2e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}}\right\rangle = ze^{x\sqrt{y}}\left\langle z\sqrt{y}, \dfrac{xz}{2\sqrt{y}}, 2\right\rangle$

44. (a) By Theorem 11.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}}f = |\nabla f|\cos\theta$ (see the proof of Theorem 11.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}f = |\nabla f|\cos\theta = \frac{1}{2}|\nabla f| \Leftrightarrow$ $\cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$.

45. $\nabla f = \langle 1/\sqrt{x}, -2y\rangle$, $\nabla f(1,5) = \langle 1, -10\rangle$, $\mathbf{u} = \frac{1}{5}\langle 3, -4\rangle$. Then $D_{\mathbf{u}}f(1,5) = \frac{43}{5}$.

46. $\nabla f = \left\langle 2xy + \sqrt{1+z}, x^2, x/\left(2\sqrt{1+z}\right)\right\rangle$, $\nabla f(1,2,3) = \left\langle 6, 1, \frac{1}{4}\right\rangle$, and $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}f(1,2,3) = \frac{25}{6}$.

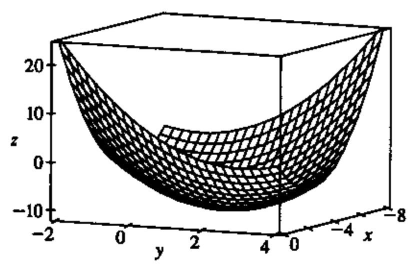
47. $\nabla f = \left\langle 2xy, x^2 + 1/\left(2\sqrt{y}\right)\right\rangle$, $|\nabla f(2,1)| = \left|\left\langle 4, \frac{9}{2}\right\rangle\right|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2}\right\rangle$.

48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle$, $\nabla f(0,1,2) = \langle 2,0,1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2,0,1\rangle| = \sqrt{5}$.

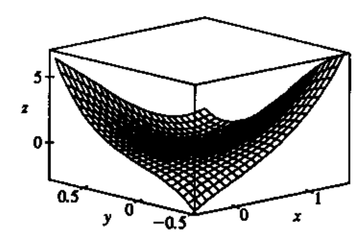
49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

50. The surfaces are $f(x,y,z) = z - 2x^2 + y^2 = 0$ and $g(x,y,z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2,2,4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x,y,z) = \langle -4x, 2y, 1\rangle \;\Rightarrow\; \nabla f(-2,2,4) = \langle 8,4,1\rangle$, $\nabla g(x,y,z) = \langle 0,0,1\rangle \;\Rightarrow$ $\nabla g\langle -2,2,4\rangle = \langle 0,0,1\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}$. Thus, parametric equations are: $x = -2 + 4t$, $y = 2 - 8t$, and $z = 4$.

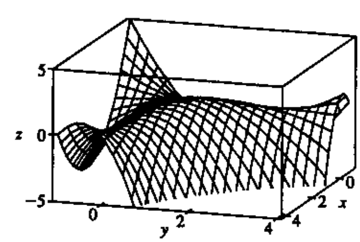
51. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \;\Rightarrow\; f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4,1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



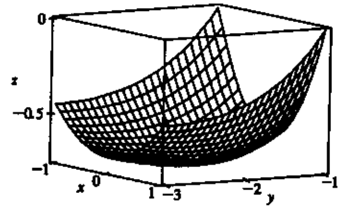
52. $f(x,y) = x^3 - 6xy + 8y^3 \;\Rightarrow\; f_x = 3x^2 - 6y$, $f_y = -6x + 24y^2$, $f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0,0)$, $(1, \frac{1}{2})$. $D(0,0) = -36 < 0$ so $(0,0)$ is a saddle point while $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and $D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local minimum.



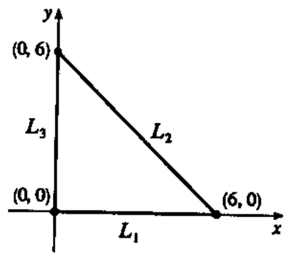
53. $f(x,y) = 3xy - x^2y - xy^2 \;\Rightarrow\; f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$. $D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



54. $f(x,y) = (x^2+y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2+x^2+y)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4+x^2+y)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0,-2) > 0$, $D(0,-2) = e^{-2} - 0 > 0$ so $f(0,-2) = -2/e$ is a local minimum.



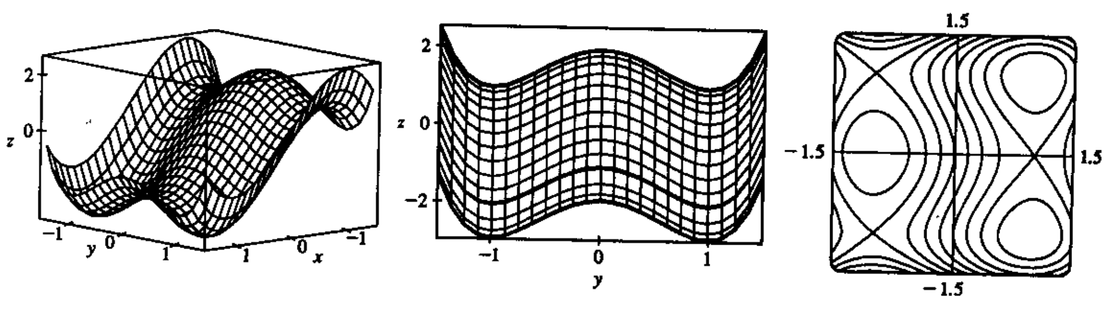
55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D. On L_1, $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2, $x = -y + 6$ and $f9 - y + 6, y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has



critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3, $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

56. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm1, 0)$. Now $f(0,0) = 0$, $f((\pm1,0) = e^{-1}$ and $f(0,\pm1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm2, 0) = 4e^{-4}$, $f(0,\pm2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0,\pm1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

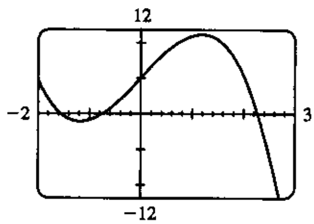
57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$

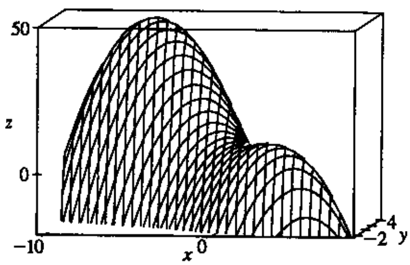


From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minima $f(1,\pm1) \approx -3$, and saddle points at $(-1,\pm1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad \Leftrightarrow \quad x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \quad \Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1, 0) = 24 > 0$ and $f_{xx}(-1, 0) = -6 < 0$ indicating a local maximum $f(-1, 0) = 2$; $D(1, \pm 1) = 48 > 0$ and $f_{xx}(1, \pm 1) = 6 > 0$ indicating local minima $f(1, \pm 1) = -3$; and $D(-1, \pm 1) = -48$ and $D(1, 0) = -24$, indicating saddle points.

58. $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4 \quad \Rightarrow \quad f_x(x, y) = -4x - 8y$, $f_y(x, y) = 10 - 8x - 4y^3$. Now $f_x(x, y) = 0 \quad \Rightarrow \quad x = -2x$, and substituting this into $f_y(x, y) = 0$ gives $10 + 16y - 4y^3 = 0 \quad \Leftrightarrow$ $5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x, y) = x^2 y$, $g(x, y) = x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0, 0) = 0$.] Thus the possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x, y) = 1/x + 1/y$, $g(x, y) = 1/x^2 + 1/y^2 = 1 \quad \Rightarrow \quad \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

61. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z} \Rightarrow$ $2y^2z = 2x^2z \Rightarrow y^2 = x^2$, and similarly $2yz^2 = 2x^2y \Rightarrow z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \Rightarrow x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1,1,\pm1)$, $(1,-1,\pm1)$, $(-1,1,\pm1)$, $(-1,-1,\pm1)$. The absolute maximum is $f(1,1,1) = f(1,-1,-1) = f(-1,1,-1) = f(-1,-1,1) = 1$ and the absolute minimum is $f(1,1,-1) = f(1,-1,1) = f(-1,1,1) = f(-1,-1,-1) = -1$.

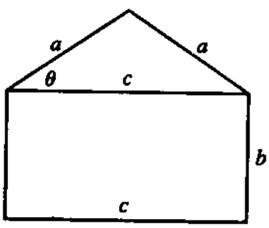
62. $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = x - y + 2z = 2 \Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and (1) $2x = \lambda + \mu$, (2) $4y = \lambda - \mu$, (3) $6z = \lambda + 2\mu$, (4) $x + y + z = 1$, (5) $x - y + 2z = 2$. Then six times (1) plus three times (2) plus two times (3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus four times (3) implies $12(x - y + 2z) = 7\lambda + 17\mu$, so (5) gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (2) and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0,0,1)$ satisfies both constraints and $f(0,0,1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

63. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = xy^2z^3 = 2 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so (1) $2x = \lambda y^2z^3$, (2) $1 = \lambda xz^3$, (3) $2 = 3\lambda xy^2z$. Then (2) and (3) imply $\frac{1}{xz^3} = \frac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly (1) and (3) imply $\frac{2x}{y^2z^3} = \frac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x,y,z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$, $\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z) = 2$ and $f(2,1,1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x,y,z) = xy^2z^3 = 2$ implies $y^2 = \frac{2}{xz^3}$, so minimize $f(x,z) = x^2 + \frac{2}{xz^3} + z^2$. Then $f_x = 2x - \frac{2}{x^2z^3}$, $f_z = -\frac{6}{xz^4} + 2z$, $f_{xx} = 2 + \frac{4}{x^3z^3}$, $f_{zz} = \frac{24}{xz^5} + 2$ and $f_{xz} = \frac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2+4)\left(2 + \frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \frac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

64. $V = xyz$, say x is the length and $x + 2y + 2z \le 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy\rangle = \langle \lambda, 2\lambda, 2\lambda\rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11{,}664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x, y, z) = 108$ and $V(104, 1, 1) = 104$ cubic units, $(36, 18, 18)$ gives an absolute maximum of V subject to $g(x, y, z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108 - \alpha)/6 = z$, $x = (108 - \alpha)/3$ with $V = (108 - \alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11{,}664$. Hence we have shown that the maximum of V subject to $g(x, y, z) \le 108$ is the maximum of V subject to $g(x, y, z) = 108$ (an intuitively obvious fact).

65.



The area of the triangle is $\frac{1}{2}ca \sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a, b, c) = \frac{1}{2}ca \sin\theta + bc$. The perimeter of the object is $g(a, b, c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2 \sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \;\Rightarrow\; \sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a, b, c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$.

(Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: (1) $ca\left(4a^2 - c^2\right)^{-1/2} = 2\lambda$, (2) $c = 2\lambda$, (3) $\frac{1}{4}\left(4a^2 - c^2\right)^{1/2} - \frac{1}{4}c^2\left(4a^2 - c^2\right)^{-1/2} + b = \lambda$, and (4) $2a + 2b + c = P$. From (2), $\lambda = \frac{1}{2}c$ and so (1) produces $ca\left(4a^2 - c^2\right)^{-1/2} = c \;\Rightarrow\; \left(4a^2 - c^2\right)^{1/2} = a \;\Rightarrow\; 4a^2 - c^2 = a^2 \;\Rightarrow\;$ (5) $c = \sqrt{3}\,a$.

Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, (3) gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from (5), $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}a}{2}$ $\Rightarrow\; -\dfrac{a}{2} - \dfrac{\sqrt{3}a}{2} = -b \;\Rightarrow\;$ (6) $b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$. Substituting (5) and (6) into (4) we get:

$$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \frac{P}{3 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{3}P \text{ and thus}$$

$$b = \frac{\left(2\sqrt{3} - 3\right)\left(1 + \sqrt{3}\right)}{6}P = \frac{3 - \sqrt{3}}{6}P \text{ and } c = \left(2 - \sqrt{3}\right)P.$$

66. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \left(f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}\right)\mathbf{k}$ (by the Chain Rule). Therefore

$$\begin{aligned} K = \tfrac{1}{2}m\,|\mathbf{v}|^2 &= \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^2\right] \\ &= \frac{m}{2}\left[\left(1 + f_x^2\right)\left(\frac{dx}{dt}\right)^2 + 2f_xf_y\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \left(1 + f_y^2\right)\left(\frac{dy}{dt}\right)^2\right] \end{aligned}$$

(b) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2x}{dt^2}\,\mathbf{i} + \dfrac{d^2y}{dt^2}\,\mathbf{j} + \left[f_{xx}\left(\dfrac{dx}{dt}\right)^2 + 2f_{xy}\,\dfrac{dx}{dt}\,\dfrac{dy}{dt} + f_{yy}\left(\dfrac{dy}{dt}\right)^2 + f_x\,\dfrac{d^2x}{dt^2} + f_y\,\dfrac{d^2y}{dt^2}\right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t\cos t$ and $y = t\sin t$, then $z = f(x,y) = t^2$.

$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} + 2t\,\mathbf{k}$,

$K = \dfrac{m}{2}\left[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2\right] = \dfrac{m}{2}(1 + t^2 + 4t^2) = \dfrac{m}{2}(1 + 5t^2)$, and

$\mathbf{a} = (-2\sin t - t\cos t)\,\mathbf{i} + (2\cos t - t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).