PRACTICE PROBABILITY PROBLEMS FOR 21A

This is a collection of problems similar to the kind of problems that will be on the exam. If a topic is not covered in this collection it does not mean you are not responsible for it. To determine the collection of topics from which the exam questions will be drawn please refer to the outline on the website.

Repeated Trials and Binomial Distribution
Pitman Sections 2.1

(1) At Princeton University, grade inflation has become so extreme that the NCAA requires athletes to get A's in at least half of their courses to remain eligible. A star football player can choose to take four courses (p=1/2 for an A) or just three courses (p=2/3 for an A). Which choice gives the higher probability of remaining eligible?

Ans: If the student takes four courses he has to get A's in two or more courses.

$$\begin{aligned}\mathbb{P}(\text{A's } \geq 2) &= \binom{4}{2} * (\tfrac{1}{2})^4 + \binom{4}{3} * (\tfrac{1}{2})^4 + \binom{4}{4} * (\tfrac{1}{2})^4 \\ &= \tfrac{11}{16}\end{aligned}$$

If he takes 3 courses he still has to get A's in two or more courses.

$$\begin{aligned}\mathbb{P}(\text{A's } \geq 2) &= \binom{3}{2} * (\tfrac{2}{3})^2(\tfrac{1}{3}) + \binom{3}{3} * (\tfrac{2}{3})^3 \\ &= \tfrac{20}{27}\end{aligned}$$

So taking 3 courses gives the higher probability of eligibility.

(2) An unprepared students takes a quiz consisting of five multiple-choice questions, each with possible answers. Knowing nothing about the subject, the student answers each question by tossing a coin twice to choose randomly among the four answers, and so has a probability of p=1/4 of getting each question correct.
a. What is the probability that the student gets precisely two correct answers?
b. What is the probability that the student gets four or more correct answers?
c. What is the most probable number of correct answers?
d. Suppose that two questions correct was enough to pass the quiz, and the student learns that he failed. What is the (conditional) probability that he had zero correct answers?

Ans: a. $\mathbb{P}(\text{Correct} = 2) = \binom{5}{2}(\frac{1}{4})^2(\frac{3}{4})^3 = \frac{135}{512}$
b. $\mathbb{P}(\text{Correct} \geq 4) = \mathbb{P}(\text{Correct} = 4) + \mathbb{P}(\text{Correct} = 5) = \binom{5}{4}(\frac{1}{4})^4(\frac{3}{4})^1 + \binom{5}{5}(\frac{1}{4})^5 = \frac{1}{4}^3 = \frac{1}{64}$
c. By calculating all the possibilities and comparing one sees that $\mathbb{P}(\text{Correct} = 1) = \frac{5}{4} * \frac{3}{4}^4$ is largest. So the most probable number of correct answers is 1.

d. By Bayes' rule,

$$\begin{aligned}
\mathbb{P}(C=0|\text{failed}) &= \frac{\mathbb{P}(\text{failed}|C=0)*\mathbb{P}(C=0)}{\mathbb{P}(\text{failed})}. \\
\mathbb{P}(\text{failed}|C=0) &= 1, \\
\mathbb{P}(C=0) &= (\frac{3}{4})^5, \text{and} \\
\mathbb{P}(\text{failed}) &= \mathbb{P}(C\leq 1) = (\frac{3}{4})^5 + 5*(\frac{3}{4})^4*\frac{1}{4} \\
&= 2*(\frac{3}{4})^4.
\end{aligned}$$

So $\mathbb{P}(C=0|\text{failed}) = \frac{1}{2}*\frac{3}{4} = \frac{3}{8}$.

(3) If a fair die is rolled four times, what is the probability that a 6 appears on precisely two of the rolls?

Ans: $\binom{4}{2}(\frac{1}{6})^2(\frac{5}{6})^2 = \frac{25}{216}$.

Bayes' Rule and Conditional Probability
Pitman Section 1.4 and 1.5

(1) In an eight-team football league there are four referees. At the start of the season coins are distributed to them at random. Three of the coins are fair ones, but the fourth has two heads. Event A is that the referee R receives the two-headed coin. An astute coach notices that the first three coin flips of referee R have all come up heads. This is event B. The coach proposes to say to the referee, "Are you using a two-headed coin?" but he wants to know the probability of event A in order to be sure that he has a good chance of being correct.

Calculate the conditional probability, given event B, that referee R is using the two-headed coin.

Ans: This is a direct application of Bayes' Rule.

$$\begin{aligned}
\mathbb{P}(\text{unfair coin}|\text{3 Heads}) &= \frac{\mathbb{P}(\text{3 H})|\text{unfair})\mathbb{P}(\text{unfair})}{\mathbb{P}(\text{3 H}|\text{fair})\mathbb{P}(\text{fair}) + \mathbb{P}(\text{3 H}|\text{unfair})\mathbb{P}(\text{unfair})} \\
&= (1*\frac{1}{4})/((\frac{1}{2})^3*\frac{3}{4}+\frac{1}{4}) \\
&= \frac{8}{11}
\end{aligned}$$

(2) (This problem is loosely based on some work with a recently-discovered 1774 census of Rhode Island)

A genealogist, on analyzing names in the 18th-century Rhode Island, has ascertained the following:

The probability that a male child was a slave was 0.3.

Male children who were slaves were given classical names like "Caesar" and "Aesop" with probability 0.6.

Male children who were free were given classical names like "Caesar" and "Aesop" with probability 0.2.

Event A is "the child was a slave" and event B is "the child had a classical name."

a. Using set-theoretic notation, express the event "the child was either free or had a classical name" in terms of events A and B and Calculate its probability.

b. Are events A and B independent? Justify your answer.

c. The genealogist encounters the classical name "Cicero Greene". What is the conditional probability, given his name, that Cicero was a slave?

Ans: a.$A \cup B = \{$the child was either free or had a classical name $\}$. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. $\mathbb{P}(A) = .3$ and

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) \\
&= \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c) \\
&= .6 * .3 + .2 * .7 = .32, \text{ but} \\
\mathbb{P}(A \cap B) &= \mathbb{P}(B|A)\mathbb{P}(A) = .6 * .3 = .18 \text{ so} \\
\mathbb{P}(A \cup B) &= .3 + .32 - .18 = .44
\end{aligned}$$

b. $\mathbb{P}(B|A) = .6$ and $\mathbb{P}(B) = .32$ so A and B are not independent.

c.

$$\begin{aligned}
\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= \frac{.18}{.32} = .5625
\end{aligned}$$

(3) Every day, a website posts a "four letter word of the day," and students have become fond of betting on its properties. A clever mathematics student hacks the website and determines the following: For "event A" (the words begin with a consonant), $\mathbb{P}(A) = 0.8$.

For "event B" (the word ends with a vowel), $\mathbb{P}(B) = 0.3$. For "event C" (the word begins

and ends with a consonant), $\mathbb{P}(C) = 0.6$. a. Using set-theoretic notation, express event

C in terms of events A and B.

b. Are events A and B independent? Justify your answer.

c.Event D is that the word has a consonant at the beginning, at the end, or both at the beginning and at the end. Express this event in terms of A and B, and calculate its probability.

d.If you know that today's word ends in a vowel, what is the conditional probability that it also begins with a vowel?

Ans: a. $C = A \cap B^c$

b.

$$\begin{aligned}
\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) &= \mathbb{P}(A) \text{ so} \\
\mathbb{P}(A \cap B) &= \mathbb{P}(A) - \mathbb{P}(A \cap B^c)) \\
&= .8 - .6 = .2 \\
\mathbb{P}(A) * \mathbb{P}(B) &= .8 * .3 = .24
\end{aligned}$$

So $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) * \mathbb{P}(B)$, thus A and B are not independent. c. $D = A \cup B^c$.

$$\begin{aligned}
\mathbb{P}(D) &= \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c) \\
&= \mathbb{P}(A) + (1 - \mathbb{P}(B)) - \mathbb{P}(C) \\
&= .8 + .7 - .6 \\
&= .9
\end{aligned}$$

d.

$$\begin{aligned}
\mathbb{P}(A^c|B) &= \frac{\mathbb{P}(B \cap A^c)}{\mathbb{P}(B)} \\
&= \frac{1 - \mathbb{P}(A \cup B^c)}{\mathbb{P}(B)} = \frac{1 - \mathbb{P}(D)}{\mathbb{P}(B)} \\
&= \frac{.1}{.3} = \frac{1}{3}
\end{aligned}$$

(4) A chance device used by the Lottery Commission can generate any number between 2 and 30 for the "daily number game", with the probability of any individual number determined by secret formula.

Event A is "the number is prime" and $\mathbb{P}(A) = 0.4$.
Event B is "the number is less than 15" and $\mathbb{P}(B) = 0.5$.
Event C is "the number is a prime less than 15" and $\mathbb{P}(C) = 0.3$.

a. Are events A and B independent? Explain.

For each of the following events, specify the event in terms of A and B, and calculate its probability.
b. Event D: "the number us a prime greater than or equal to 15."
c. Event E: "the number is either prime or less than 15."

Suppose that today's number has been generated. d. If it is known to be prime, what is the probability that it is also less than 15?
e. If it is known to be greater than or equal to 15, what is the probability that it is not prime?

Ans: a. $A \cap B = C$ so $\mathbb{P}(A \cap C) = 0.3$ but $\mathbb{P}(A) * \mathbb{P}(B) = 0.2$ so A and B are not independent.
b. $D = A \cap B^c$ and

$$\begin{aligned}
\mathbb{P}(D) &= \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \\
&= .4 - .3 = .1
\end{aligned}$$

c. $E = A \cup B$.

$$\mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = .4 + .5 - .3 = .6$$

d.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{.3}{.4} = \frac{3}{4}$$

e.

$$\begin{aligned}\mathbb{P}(A^c|B^c) &= \frac{\mathbb{P}(A^c \cap B^c)}{\mathbb{P}(B^c)} = \frac{1-\mathbb{P}(A\cup B)}{1-\mathbb{P}(B)} \\ &= \frac{1-\mathbb{P}(E)}{1-\mathbb{P}(B)} = \frac{1-.6}{1-.5} = \frac{4}{5}\end{aligned}$$

(5) A gambler owns three dice. Two of them are unloaded, but the third has $p = 1/2$ for a six, and $p = 1/10$ for the numbers one through five. The gambler chooses one of the dice, each with equal probability, and then rolls the chosen die three times.

Event A is "the gambler choose the loaded die."

Event B is "Of the three rolls, exactly 1 is a six."

a. Calculate $\mathbb{P}(A \cap B)$, $\mathbb{P}(A^c \cap B)$, and $\mathbb{P}(B)$.

b. Given the event B, what is the conditional probability that the gambler is using the loaded die?

Ans: a.

$$\mathbb{P}(A) = \frac{1}{3}$$

$$\mathbb{P}(B|A) = \binom{3}{1}(\frac{1}{2})^1(\frac{1}{2})^2 = \frac{3}{8}$$

$$\mathbb{P}(B|A^c) = \binom{3}{1}(\frac{1}{6})(\frac{5}{6})^2 = \frac{25}{72}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = \frac{1}{8}$$

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B|A^c)\mathbb{P}(A^c) = \frac{25}{108}$$

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) = \frac{1}{8} + \frac{25}{108} = \frac{77}{216}$$

b.

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} \\ &= \frac{1}{8} * \frac{216}{77} = \frac{27}{77}\end{aligned}$$

Continuous Random Variables

Pitman 4.1 and Stewart 12.5

(1) A two-dimensional random variable (X, Y) has a probability density function of the form:

$f(x, y) = 0$ if $x < 0$ or $y < 0$.

$f(x, y) = 0$ if $x > 3$ or $y > 2$.

$f(x, y) = cx^2y$ otherwise.

a. Determine what value the constant c must have.

b. Calculate the probability of the event $X^2 + Y^2 \leq 1$.

c. Calculate the expectation of $X + Y$.

a.

$$\int_0^2 \int_0^3 cx^2 y dx dy = 1$$

So $c = \frac{1}{18}$.
b.

$$\mathbb{P}(X^2 + Y^2 \leq 1) = \int\int_{\{x^2+y^2 \leq 1\}} f(x,y) dx dy$$

By changing to polar coordinates we get

$$\mathbb{P}(X^2 + Y^2 \leq 1) = \int_0^{\pi/2} \int_0^1 \frac{1}{18} * (r^2 \cos^2 \theta)(r \sin \theta) r dr d\theta$$

Thus,

$$\mathbb{P}(X^2 + Y^2 \leq 1) = \frac{1}{270}$$

c.

$$\mathbb{E}(X+Y) = \int_0^2 \int_0^3 (x+y) f(x,y) dx dy = \frac{43}{6}$$

(2) Suppose that the random variable X is uniformly distributed in the interval (1,27] and that the random variable Y is given by the function $Y = \frac{1}{3} * X$.
a. What is the probability of the event $2 < Y \leq 3$?
b. Determine the probability density function $f_Y(y)$ for the variable Y.
c. What is the expectation of Y?

Ans: a. $\mathbb{P}(2 < Y \leq 3) = \mathbb{P}(2 < \frac{1}{3} * X \leq 3)$.
$\mathbb{P}(2 < \frac{1}{3} * X \leq 3) = \mathbb{P}(6 < X \leq 9) = \frac{3}{26}$.
b.
$f_Y(y) = \frac{26}{9}$ for $1/3 \leq y \leq 9$ and $f_Y(y) = 0$ otherwise.
c. $\mathbb{E}(Y) = 14/3$.

(3) a. Is $f(x,y) = 4xy$ if $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $f(x,y) = 0$ otherwise a joint density function? Why?
b. If X and Y are random variables whose joint density function is $f(x,y)$ from part (a), find $\mathbb{P}(X \geq \frac{1}{2})$ and $\mathbb{P}(X \geq \frac{1}{2}, Y \leq \frac{1}{2})$.
c. Find the expected values of X and Y.

Ans:
a. Yes, because it has integral equal to one (on an exam you would have to show the calculation).
b. $\mathbb{P}(X \geq 1/2) = \int_{1/2}^1 \int_0^1 4xy dx dy = 3/4$.
$\mathbb{P}(X \geq 1/2, Y \leq 1/2) = \int_{1/2}^1 \int_0^{1/2} 4xy dx dy = 3/16$.
c. $\mathbb{E}(X) = \int_0^1 \int_0^1 x * 4xy dx dy = 2/3$ and
$\mathbb{E}(Y) = \int_0^1 \int_0^1 y * 4xy dx dy = 2/3$.

(4) A two-dimensional random variable (X, Y) has probability density function of two variables given by

$$f(x,y) = 0 \text{ if } x < 0, y < 0, \text{ or } x + y > 2$$

$$f(x,y) = cxy \text{ otherwise}$$

a. Sketch the region where $f(x,y) > 0$.
b. Determine the constant c must have.
c. Calculate the probability of the event $X < 1$.

Ans: b. Solve the equation:

$$\int_0^2 \int_0^{2-x} cxydydx = 1$$

This implies $c = 3/2$
c.

$$\mathbb{P}(X < 1) = \int_0^1 \int_0^{2-x} \frac{3}{2} * xydydx = \frac{11}{36}$$

Miscellaneous

(1) True-False a. For any two events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.
b.The number of distinct *four of a kind* poker hands is 13 times 48.
c. Two dice are rolled, and the sum is 6. The probability that one die or the other shows a 2 is 0.4.
d. A die is rolled seven times in succession. The most probable number of sixes is 1.
e. Suppose the (X,Y) is a two dimensional random variable, and $F(s,t) = \mathbb{P}(X < s, Y < t)$. Then $\mathbb{P}(a < X < b, c < Y < d) = F(b,d) - F(a,c)$.

Ans: a. False. b. True c. False d. True e. False

(2) The latest form of Swatch digital watch has a patented *random alarm* feature. After you push a button, the alarm goes off at a random time in the future. Consumers notice that this alarm often goes off an interal number of minutes after the alarm is pressed. One industry analyst publishes her hypothesis: namely, that the probability that the alarm goes off after precisely n minutes is equal to $n/10$. Prove that she is wrong.

Ans: The total probability of all possible outcomes must add up to 1. The hypothesis of the analyst implies that the probability of all the possible outcomes is infinite. Not only is this meaningless, but it's wrong too.