Name: ______________________________

Math 21a Final Exam – Thursday, May 16th, 2002

*Please circle your section: **Note this version is just for the Regular and Physics Sections***

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Question	Points	Score
1	8	
2	8	
3	8	
4	16	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

You have three hours to take this final exam. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order - you should move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers will receive little or no credit! Please show all of your work and be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one page (standard 8 and a half by 11 inch) of notes during the test, but you are not allowed to use any other references or calculators during this test.

Good luck! Focus and do well!

Question 1. (8 points total)

(a) Suppose that the diagonals of a certain parallelogram are perpendicular to each other. By considering the diagonals of the parallelogram as vectors, and taking their dot product, determine the relationship between the lengths of the sides of the parallelogram.

(b) What is the relationship between the cross product of the two diagonals, again considered as vectors, and the area of the parallelogram described in part (a)? Does this relationship hold for any given parallelogram, not just one whose diagonals are perpendicular?

Question 2. (8 points total)

(a) Give a set of parametric equations for the line that lies in the planes $2x+3y-z=2$ and $-x+2y+2z=4$.

(b) What is the angle between the two planes in part (a)? (recall here angle means the acute angle between the planes' normal vectors)

Question 3. (8 points total)

(a) Suppose that at the point (4, 5, 6) the function $f(x,y,z)$ increases most rapidly in the direction $\langle 4,0,-3\rangle$ and that the rate of increase of $f(x,y,z)$ in this direction is equal to 7. What is the rate of increase of $f(x,y,z)$ at the point (4, 5, 6) in the direction $\langle -1,2,-2\rangle$?

(b) Let S be the level surface of $f(x,y,z)$ that goes through the point (4, 5, 6) (where $f(x,y,z)$ is the same function as in part (a)). Write down an equation for the plane that is tangent to S at the point (4, 5, 6).

Question 4. (16 points total)

Consider the function $f(x, y) = x^2 + y^2 + 2y - 1$

(a) Find and classify all critical points of $f(x, y)$ in the plane.

(b) Consider restricting $f(x, y)$ to the constraint curve $y = x^2 + 1$. Find the extreme points of $f(x, y)$ on this curve using the Lagrange multiplier method. You don't need to classify them as minimums or maximums at this point. Note you might want to do part (d) before parts (b) and (c).

Question 4 continued

(c) Still restricting $f(x,y)=x^2+y^2+2y-1$ to the constraint curve $y=x^2+1$, now find any extreme points by parametrizing the curve (in doing this you can check your work in part (b)). Again, you can wait to classify them until you do part (d).

(d) Now sketch the constraint curve and several of the level curves of $f(x,y)$, explain your answers to parts (a), (b) and (c) using the picture, and finally classify any extreme points you found in (b) and (c) as either maximums or minimums.

Question 5. (10 points total)

It's hot, and this time you're not in a "dessert," but you are in fact eating one. Suppose you are eating an ice cream cone, whose shape (including both the cone and the ice cream in it) can be described as the region above the surface $z^2 = x^2 + y^2$ and below a sphere of radius 4 centered at the origin (if you sketch this you'll see that it's a pretty fat cone!) Suppose the density of the ice cream (along with the cone) varies depending on the distance *d* from the bottom tip of the cone, given by $\sigma = k(10 - d)$, where *k* is a constant. Find the mass of the ice cream cone (note that mass is just $\iiint_E (density)\,dV$ where *E* is the region described above).

Question 6. (10 points total)

(a) Let $f(x,y,z)$ be a potential function for a conservative vector field **F**, i.e. $\mathbf{F} = \nabla f$. Consider a level surface M for the function $f(x,y,z)$, so that M is given by $f(x,y,z) = k$, for some constant k (such a surface is called an equipotential surface). If C is a curve (not necessarily closed) on such a surface M, then explain why it is that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

(b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle -e^y \sin(x), e^y \cos(x) \rangle$ and C is the curve given by

$\mathbf{r}(t) = \langle \pi - \pi\cos(t), \pi\sin(t) \rangle$ with $0 \le t \le \pi$

Question 7. (10 points total)

Use Green's Theorem and the vector field $\mathbf{F} = \langle 0, x^3 y \rangle$ to compute the double integral

$\iint_R (3x^2 y)\,dA$ where R is the region inside the ellipse $x^2 + \frac{y^2}{4} = 1$

For the boundary of the ellipse note that $(\cos(t))^2 + \frac{(2\sin(t))^2}{4} = 1$

Question 8. (10 points total)

Let S be the surface given by $z = x^2 - y^2$. Let C be the curve on the surface S where $x^2 + y^2 = 1$, oriented counter-clockwise as one looks down the z-axis. Use Stokes' Theorem to calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field $\langle x^2 + z^2, y, z \rangle$

Question 9. (10 points total)

In appropriate units, the charge density $\sigma(x, y, z)$ in a region in space is given by $\sigma = \nabla \cdot \mathbf{E} = \text{div}(\mathbf{E})$ where **E** is the electric field. *(Note, you don't need to know any physics to answer this problem!)* Consider the unit cube located at the origin (the region given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$). What is the total charge in this cube if $\mathbf{E} = \left\langle x(1-x)\log(1+xyz), y(1-y)\tan(xyz), z(1-z)e^{xyz} \right\rangle$? (note, the total charge in a region is the integral of the charge density over the region)

Be sure to show all your work and explain your reasoning to receive full credit for your answer.

Regular and Biochem sections answer Question 10A, Physics section answer Question 10B

Question 10A (Regular and Biochem sections). (10 points total)

(i) Let g(u) be a differentiable function and let $f(x,y) = g(x^2 + y^2)$. Show that $f(x,y)$ is a solution to the partial differential equation $y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = 0$

(ii) Recall that the general solution to the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = u(l,t) = 0$ for all t is given by

$$u(x,t) = \left(A\cos(\lambda_n t) + B\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

where A and B are constants, n is an integer, and $\lambda_n = \frac{cn\pi}{l}$.

Find *one* example (there are infinitely many) of a specific solution to the wave equation $\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = u(2\pi,t) = 0$ and initial condition $u(x,0) = 0$ (note, yes, the function $u(x,t) = 0$ is a solution, but please find a different one!)

Regular and Biochem sections answer Question 10A, Physics section answer Question 10B

Question 10B (Physics Section). (10 points total)

(a) Let M be a *solid* sphere of radius R and constant charge density ρ. Then $Q = \frac{4}{3}\pi R^3 \rho$ is the total charge. Using Gauss's Law in integral form, and symmetry arguments, find the electric field at a distance z from the center of the sphere. Write your answer in terms of Q instead of ρ. How does the field compare to that of a point charge?

(Hint: the answer depends on if $z < R$ or $z > R$. This is not the same problem that was done in class! That was specifically for a *hollow* sphere).