Name: Answer Key

Math 21a Midterm 1 Tuesday, March 12th, 2002

Please circle your section:

Melissa Liu Roger Hong (CA) MWF 9-10	Christophe Cornut David Shim (CA) MWF 10-11	Spiro Karigiannis Michael Simonetti (CA) MWF 10-11
Spiro Karigiannis Gloria Hou (CA) MWF 11-12	Andy Engelward Nathan Moore (CA) T/Th 10-11:30	Andy Engelward Alexey Gorshkov (CA) T/Th 11:30-12

Question	Points	Score
1	12	12
2	16	16
3	18	18
4	8	8
5	15	15
6	15	15
7	16	16
Total	100	100

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order - you should move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers will receive little or no credit! Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one 3 by 5 inch file card with formulas on it during the test, but you are not allowed to use any other notes, or calculators during this test.

Good luck! Focus and do well!

Question 1. (12 points total)

Let $f(x, y) = \ln(2x + y)$.

(a) (4 points) What is the domain and range of $f(x, y)$?

Domain: for $\ln(u)$, u must be positive, so domain is just $2x+y>0$, or $y>-2x$

Range: output from $\ln(u)$ is all $\mathbb{R}$, or $(-\infty, \infty)$

(b) (4 points) Describe the level curves for the graph of $z = f(x, y)$ as accurately as possible.

Solving $f(x,y)=k$, then $\ln(2x+y)=k$, $2x+y=e^k$ so the $z=k$ level curve is a straight line $y=-2x+e^k$ with slope -2. Since all the level curves are lines with the same slope then the contour map for $f(x,y)=\ln(2x+y)$ is a series of parallel lines of slope -2.

(c) (4 points) Find the coordinates of the point where the $z = 0$ level curve for the function $f(x, y)$ intersects the y-axis.

Set $0=f(x,y)=\ln(2x+y)$ to find the $z=0$ level curve. Where this intersects the y-axis the $\underline{\underline{x}}$ coordinate $=0$, so $0=\ln(0+y)$, $e^0=1=y$, so the coordinates of the point in question are $(0,1)$ (Note level curves live in the xy plane so this point has just the two coordinates)

Question 2. (16 points total)

(a) (4 points) Under what conditions is **a** x **b** = **b** x **a**?

Since $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$, then the statement implies that $-\vec{b}\times\vec{a} = \vec{b}\times\vec{a}$, which can only happen if $\vec{b}\times\vec{a} = \vec{0}$, which is true only if $\vec{a}$ and $\vec{b}$ are parallel vectors, or if at least one of $\vec{a}$ or $\vec{b}$ is equal to $\vec{0}$.

(b) (4 points) If **a** x **b** = **a** x **c**, then which of the following statements must be true? (circle all the correct answers)

(i) **b** = **c** (ii) |**b**| = |**c**| (iii) **b** and **c** are parallel

(iv) **a** must be a unit vector (v) **a** is parallel to **b** - **c**

Why? → if $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ then $\vec{a}\times(\vec{b}-\vec{c}) = 0$ which implies (v). None of the other answers, such as (i) must be true, however. (If $\vec{b}=\vec{c}$ then $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, but $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ does not force $\vec{b}=\vec{c}$)

(c) (4 points) If you know that |**a** x **b**| = |**a**| |**b**| then what must be true of the two vectors?

Since $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin(\theta)$ (θ angle between $\vec{a}, \vec{b}$)
and $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|$, then $\sin(\theta) = 1$,
or $\theta = 90°$, so $\vec{a}$ and $\vec{b}$ must be perpendicular

(d) (4 points) For what values of t are **a** = $(t+2)$**i** + t**j** + t**k** and **b** = $(t-2)$**i** + $(t+1)$**j** + t**k** orthogonal?

Orthogonal implies that $\vec{a}\cdot\vec{b} = 0$, and since
$\vec{a}\cdot\vec{b} = (t+2)(t-2) + t(t+1) + t\cdot t = t^2 - 4 + t^2 + t + t^2$
$= 3t^2 + t - 4$, then $\vec{a}$ and $\vec{b}$ will be orthogonal precisely when $3t^2+t-4=0$,
or $(3t+4)(t-1) = 0$, so when $t = 1$ or $-\frac{4}{3}$

Question 3. (18 points total)

Consider the parametrized curve defined by the vector function
$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), 2t \rangle$ with $0 \le t \le 3\pi$

(Note, parts (a) and (b) are almost identical to homework problem 10.1 #19)

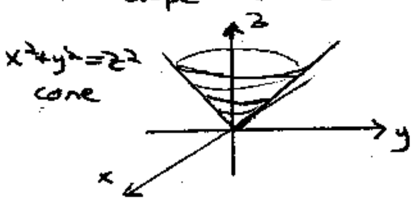
(a) (3 points) Find an equation for a surface that this space curve lies on and describe the surface (hint, the curve is an example of a conical helix)

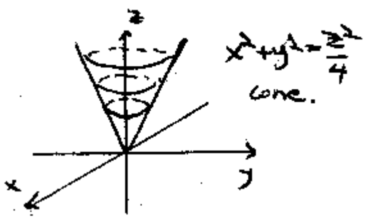
Conical helix hint... possibly look for something of form $\frac{x^2}{a} + \frac{y^2}{b} = \frac{z^2}{c}$, here $x^2+y^2 = (t\cos(t))^2 + (t\sin(t))^2 = t^2$ and $z^2 = (2t)^2 = 4t^2$, so $x^2+y^2 = \frac{z^2}{4}$ or $4x^2+4y^2 = z^2$

This is indeed a cone

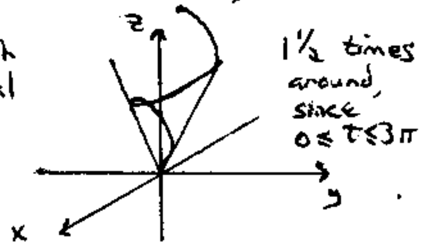
(b) (3 points) Sketch the curve as neatly as possible.

The cone $x^2+y^2 = \frac{z^2}{4}$ looks just like $x^2+y^2=z^2$ cone, but steeper sided: when $y=0$, $x^2 = \frac{z^2}{4}$, or $z = \pm 2x$, a slope 2 line:





Now with the conical helix:



(c) (4 points) Find the length of this space curve by giving the length as a definite integral (you do not need to evaluate the integral!) $\vec{r}\,'(t) = \langle -t\sin(t)+\cos(t), t\cos(t)+\sin(t), 2 \rangle$

Then arc length is just:

$$\int_0^{3\pi} |\vec{r}\,'(t)|\,dt = \int_0^{3\pi} \sqrt{(-t\sin(t)+\cos t)^2 + (t\cos t + \sin t)^2 + 2^2}\,dt = \int_0^{3\pi} \sqrt{t^2+5}\,dt$$

(d) (4 points) Give parametric equations for the tangent line to this curve at the point corresponding to $t = 2\pi$. $\vec{r}(2\pi) = \langle 2\pi, 0, 4\pi \rangle$ $\quad \vec{r}\,'(2\pi) = \langle 1, 2\pi, 2 \rangle$,

so the tangent line is given by $\vec{r} + s\vec{r}\,' = \langle 2\pi, 0, 4\pi \rangle + s\langle 1, 2\pi, 2 \rangle$

or $= \langle 2\pi + s, 0 + s2\pi, 4\pi + 2s \rangle$, with parametric equations:

$$x(s) = 2\pi + s, \quad y(s) = 2\pi s, \quad z(s) = 4\pi + 2s$$

(e) (4 points) At what point does the tangent line from part (d) intersect the plane $\pi x + y + \pi z = 11\pi^2$?

When does $\langle 2\pi + s, 2\pi s, 4\pi + 2s \rangle$ (the line from part (d)) intersect $\pi x + y + \pi z = 11\pi^2$? $\rightarrow$ when the components satisfy the equation, i.e. $\pi x + y + \pi z$

$= \pi(2\pi + s) + 2\pi s + \pi(4\pi + 2s) = 2\pi^2 + \pi s + 2\pi s + 4\pi^2 + 2\pi s$

$= 6\pi^2 + 5\pi s = 11\pi^2$ when $s = \pi$,

so at the point $\langle 2\pi + \pi, 2\pi^2, 4\pi + 2\pi \rangle = \langle 3\pi, 2\pi^2, 6\pi \rangle$

Question 4. (8 points total)

(a) (4 points) Using the product rule (twice), work out an expression for $\frac{d}{dt}\left[\left(\vec{u}(t)\times\vec{v}(t)\right)\cdot\vec{w}(t)\right]$

(note rewriting just "$\frac{d}{dt}\left[\left(\vec{u}(t)\times\vec{v}(t)\right)\cdot\vec{w}(t)\right]$" does not count!)

We know $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}(t)\times\vec{v}'(t) + \vec{u}'(t)\times\vec{v}(t)$

and $\frac{d}{dt}[\vec{m}(t)\cdot\vec{w}(t)] = \vec{m}(t)\cdot\vec{w}'(t) + \vec{m}'(t)\cdot\vec{w}(t)$

so putting these together:

$$\frac{d}{dt}\left[\left(\vec{u}(t)\times\vec{v}(t)\right)\cdot\vec{w}(t)\right] = \left(\vec{u}(t)\times\vec{v}(t)\right)\cdot\vec{w}'(t) + \left[\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right)\right]\cdot\vec{w}(t)$$

$$= \left(\vec{u}(t)\times\vec{v}(t)\right)\cdot\vec{w}'(t) + \left(\vec{u}(t)\times\vec{v}'(t) + \vec{u}'(t)\times\vec{v}(t)\right)\cdot\vec{w}(t)$$

(b) (4 points) Find the points on the space curve $\mathbf{r}(t) = \langle t^2 + 2t, 3, t^3 - 12t \rangle$ where the tangent vector is parallel to one of the standard basis vectors, **i**, **j**, or **k**.

$\vec{r}'(t) = \langle 2t+2, 0, 3t^2-12 \rangle$. To be parallel to either $\vec{i}$, $\vec{j}$ or $\vec{k}$ means to be a multiple of one of them, i.e. in the form $m\vec{i} = \langle m,0,0\rangle$

or $m\vec{j} = \langle 0,m,0\rangle$ or $m\vec{k} = \langle 0,0,m\rangle$

Thus either $3t^2 - 12 = 0 \Rightarrow t = \pm 2$

or $2t+2 = 0 \Rightarrow t = -1$

Then the 3 points on the space curve are thus

when $t = +2$ $\quad \langle 4+4, 3, 8-24\rangle = \langle 8, 3, -16\rangle$

" $t = -2$ $\quad \langle 4-4, 3, -8+24\rangle = \langle 0, 3, 16\rangle$

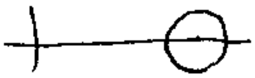
" $t = -1$ $\quad \langle 1-2, 3, -1+12\rangle = \langle -1, 3, 11\rangle$

Question 5. (15 points total)

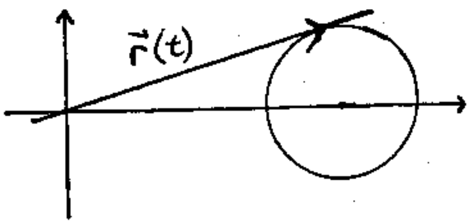
(a) (4 points) Find a parametrization for the circle in the xy-plane of radius 1, centered at (5,0).

parametrization for unit circle  is just

$x = \cos t, \quad y = \sin t,$

so for  unit circle moved over 5 in x direction we can use $x(t) = 5 + \cos t$, $y(t) = \sin(t)$

(b) (11 points) Find an equation for the line of positive slope that passes through the origin (0,0) and which is tangent to the circle of radius 1 with center at (5,0) (hint, use your parametrization from part (a)).



There are many ways to approach this. Using the hint we know that $\vec{r}(t) = \langle 5+\cos t, \sin t\rangle$ is a parametrization of the circle in question, with $\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$

So the tangent line at any point on the circle is

$\vec{r}(t) + s\,\vec{r}\,'(t)$ (Note! you need to use different variable names here as t is the parameter giving position on circle, and s is the parameter giving position on tangent line)

$= \langle 5+\cos t, \sin t\rangle + s\langle -\sin t, \cos t\rangle$

$= \langle 5+\cos t - s\cdot\sin t, \ \sin t + s\cdot\cos t\rangle$

So now the question is for what value of t does this tangent line go through the origin?

$\langle 5+\cos t - s\cdot\sin t, \ \sin t + s\cdot\cos t\rangle = \langle 0, 0\rangle$

so solving: $\sin t + s.\cos t = 0$, or $s = -\frac{\sin t}{\cos t}$, (from equating y-components)

then $5+\cos t - s.\sin t = 0$ (x-components) yields $5+\cos t - \left(\frac{-\sin t}{\cos t}\right)\sin t = 0$

or $5\cos t + \cos^2 t + \sin^2 t = 0 \Rightarrow 5\cos t + 1 = 0$ so $\cos t = -\frac{1}{5}$

Stop now $\rightarrow$ no need to solve for t, as the line goes through $(0,0)$ and $\vec{r}(t) = \langle 5+\cos t, \sin t\rangle$, so just find $\sin t$

$= \sqrt{1-\cos^2 t} = \sqrt{1-\frac{1}{25}} = \frac{\sqrt{24}}{5}$, and so the line goes through $\left(5+\left(-\frac{1}{5}\right), \frac{\sqrt{24}}{5}\right)$

slope $= \frac{\sqrt{24}}{5} \Big/ (24/5) = \frac{\sqrt{24}}{24} = \frac{1}{\sqrt{24}}$, so $\boxed{y = \frac{1}{\sqrt{24}}x}$

Question 6. (15 points total)

Let S be the surface generated by rotating the parabola $z = \frac{1}{2}\left(1 - x^2\right)$ in the xz-plane around the z-axis.

(a) (3 points) Write down an equation for the surface in cylindrical coordinates.

Note in the xz plane $x = r\cos\theta$ with $\theta = 0$, $\cos\theta = 1$, so $x = r$ (think about it $\rightarrow$ both x and r just measure the distance from the z axis, in the xz plane). So in the xz plane the equation is $z = \frac{1}{2}(1 - r^2)$, but then this is the equation for the surface generated by rotating around the z axis too.

(b) (3 points) Write down an equation for the surface in cartesian (i.e. "rectangular") coordinates.

Use standard $x = r\cos\theta$ $y = r\sin\theta$, $r^2 = x^2 + y^2$ formulas for conversion: $z = \frac{1}{2}\left(1 - (x^2 + y^2)\right)$

(c) (3 points) Write down an equation for the surface in spherical coordinates.

again just use standard conversions:

$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$

so in spherical coordinates $\rho\cos\phi = \frac{1}{2}\left(1 - \left(\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi\right)\right)$

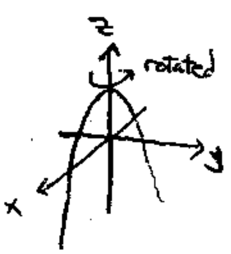
or $\rho\cos\phi = \frac{1}{2}\left(1 - \rho^2\sin^2\phi\right)$

(d) (6 points) Write down two distinct parametrizations for the surface S.

For a function $z = f(x,y)$, there is always the $(x, y, f(x,y))$ parametrization, or with $x = s$, $y = t$ you can write this as $\left(s, t, \frac{1}{2}\left(1 - (s^2 + t^2)\right)\right)$

from part (b)

For a surface formed by rotation around an axis one has the $(r(z)\cos\theta, r(z)\sin\theta, z)$ type of parametrization (see page 740).



Here when $z = \frac{1}{2}(1 - x^2)$, $2z - 1 = -x^2$, $\pm\sqrt{1 - 2z} = x$, so solving for x in terms of z gives the radius at different values of z, and gives the parametrization $\left(\sqrt{1-2t}\cos(\theta), \sqrt{1-2t}\sin(\theta), t\right)$ for $0 \le \theta \le 2\pi$, $t \le \frac{1}{2}$

Question 7. (16 points total)

Suppose there is a gas tank in the shape of a paraboloid $z = x^2 + y^2$ in your backyard (with $0 \le z \le 4$ and such that its bottom point rests at the origin (0, 0, 0)), and there is a barbecue fire going at the point Q = (1, 1, 0) (the coordinates x, y and z all measuring yards). The gas tank will blow up if the fire is closer than 1 yard to the tank. By finding the exact distance from the point (1, 1, 0) to the paraboloid, determine whether your gas tank is in serious trouble.

Two crucial pieces of information that you'll probably need to solve this death-defying problem:

(1) The straight line from the closest point on the paraboloid to the point (1, 1, 0) is in fact perpendicular to the tangent plane to the paraboloid at that closest point, and

(2) You'll probably need to use one of the following cubic factorizations at some point:

$4t^3 + 4t^2 + t - 2$ factorizes as $(2t-1)(2t^2+3t+2)$ with only one real root t = 1/2
$4t^3 - 4t^2 + t + 2$ factorizes as $(2t+1)(2t^2-3t+2)$ with only one real root t = -1/2
$4t^3 + t - 1$ factorizes as $(2t-1)(2t^2+t+1)$ with only one real root t = 1/2

Here's a challenging question! If you didn't get too far with this one, don't panic – only a small number solved it completely. However, you should have been able to at least do the computations suggested by hint #1 – "the straight line from closest point... perp to tangent plane..."
So if $(a,b,c) = (a,b,a^2+b^2)$ is a point on the paraboloid then the tangent plane through this point is given by $z - c = 2a(x-a) + 2b(y-b)$ or $2a(x-a) + 2b(y-b) - (z-c) = 0$. A normal vector to this plane is just $\langle 2a, 2b, -1\rangle$ (check your equations for planes)
Now if (a,b,c) is the closest point, then according to hint (1), the straight line is the line given by $\langle a, b, a^2+b^2\rangle + t\langle 2a, 2b, -1\rangle$ (i.e. the line through (a,b,c) perpendicular to the tangent plane).
Aha! This line supposedly goes through our point (1,1,0), so equating components, this means ① $a + 2at = 1$, ② $b + 2bt = 1$, and ③ $a^2+b^2-t=0$.
Now it all looks a bit like Lagrange multiplier method → find a, b satisfying these equations... so $a^2+b^2 = t$ from ③, but also $a = \frac{1}{1+2t}$ from ① and $b = \frac{1}{1+2t}$ from ②, so subbing in for a, b in $a^2+b^2 = t$ yields $\left(\frac{1}{1+2t}\right)^2 + \left(\frac{1}{1+2t}\right)^2 = t$, or $2 = (1+2t)^2 t$
Multiplying this out yields $4t^3 + 4t^2 + t - 2 = 0$, which according to hint (2) has real root $t = 1/2$, corresponding to the point $a = \frac{1}{1+2t} = \frac{1}{1+1} = \frac{1}{2} = b$ as well, and $c = a^2+b^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$,
i.e. the closest point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, which is a distance $\sqrt{(1-\frac{1}{2})^2 + (1-\frac{1}{2})^2 + (0-\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$ from (1,1,0). Since $\frac{\sqrt{3}}{2} < 1$ the whole thing blows up!