Name: Answer Sheet

Math 21a Midterm 2 **Wednesday, November 19th 2003**

Please circle your section:

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Jonathan Kaplan Chris Phillips MWF 11–12	Vivek Mohta Kory Byrns MWF 11–12	Vivek Mohta Ann Lai MWF 12–1	
Abhinav Kumar Justin Ho Tu/Th 10–11:30	Sabin Cautis Shih En Lu Tu/Th 10–11:30	Andy Engelward Brad Burns T/Th 11:30–1	

Question	Points	Score
1	12	
2	16	
3	16	
4	8	
5	8	
6	14	
7	14	
8	12	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order – move on to another problem if you find that you're stuck and are spending too much time on one problem.

To receive full credit on a problem, you need to justify your answers carefully (unless the question specifically says otherwise). Unsubstantiated answers will receive little or no credit! Please be sure to write neatly – illegible answers will also receive little or no credit. If you're asked to provide a sketch of something, do so as neatly as possible – you could lose points if your sketch is difficult to understand.

If you need more space for a problem then use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

Please note that you are not allowed to use any notes or calculators during this test.

Good luck! Focus and do well!

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Question 1: (12 points total)

Suppose $F(x,y) = 2e^{x-y} + 2$

(a) (7 points) Write down an equation for the tangent plane to the graph of $F(x,y)$ at the point where $x = y = 10$

We know the tangent plane has equation $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

so here $f_x = 2e^{x-y}$ and $f_y = -2e^{x-y}$

so $F_x(10,10) = 2$ $F_y(10,10) = -2$,

and $z_0 = F(10,10) = 2e^0 + 2 = 4$

Thus tangent plane equation:

$$z - 4 = 2(x-10) + (-2)(y-10)$$

or $z = 2x - 2y + 4$

(b) (5 points) Estimate the value of $F(10.1,\ 10.2)$ to one decimal place using the technique of linear approximation.

This simply means using the tangent plane at the point $(10, 10, F(10,10))$ to approximate $(10.1, 10.2, F(10.1, 10.2))$

or just sub in $x = 10.1$, $y = 10.2$ into the tangent plane from (a):

$$F(10.1, 10.2) \approx 2(10.1) - 2(10.2) + 4 = 3.8$$

Question 2. (16 points total)

Consider the function $F(x,y) = x^2 + y^2 + x^2 y$

Critical Points where $F_x = F_y = 0$ ~~(since both exist everywhere)~~

(a) (6 points) Find all of the critical points of the function $F(x,y)$.

$F_x = 2x + 2xy$ and $F_y = 2y + x^2$

so $F_x = 0$ when $x = 0$ or $y = -1$ $\quad (F_x = 2x(1+y))$

if $x = 0$ then for $F_y = 0$ implies $y = 0$ too

so one C.P. is $(0,0)$

if $y = -1$, then $F_y = -2 + x^2$ implies $x = \pm\sqrt{2}$,

so 3 C.P. points: $(0,0)$, $(\sqrt{2},-1)$, $(-\sqrt{2},-1)$

(b) (4 points) For each of the critical points you found in part (a), decide whether it is a local maximum, local minimum or a saddle point for $F(x,y)$.

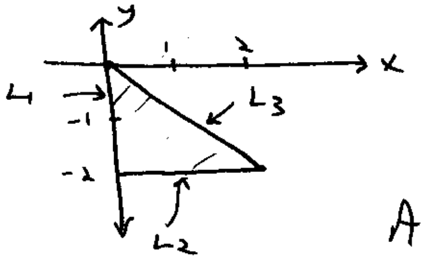
Find $D = F_{xx} \cdot F_{yy} - (F_{xy})^2 = (2+2y)\cdot(2) - (2x)^2$

at $(0,0)$ $D = 4 - 0 > 0$, and $F_{xx} = 2 > 0$
$\Rightarrow$ local min.

at $(\sqrt{2},-1)$ $D = 0 \cdot 2 - (2\sqrt{2})^2 < 0 \Rightarrow$ saddle point

same at $(-\sqrt{2},-1)$ $D < 0 \Rightarrow$ saddle point

(c) (6 points) Find the maximum value of $F(x,y)$ on the triangular region with vertices (0, 0), (0,-2) and (2,-2) (the region includes the boundary of the triangle).



There are no critical points inside the triangle, so max will occur on boundary somewhere.

Along L_1 $x = 0$, and $F(0,y) = y^2$
maximum when $y = -2$ at $(0,-2)$
where $F(0,-2) = 0 + 2^2 + 0 = 4$

Along L_2 $y = -2$, and $F(x,-2) = x^2 + 4 - 2x^2$
$= -x^2 + 4$

max. occurs when $x = 0$, at $(0,-2)$ again.

Along L_3 $y = -x$, so $F(x,-x) = x^2 + (-x)^2 + x^2(-x) = 2x^2 - x^3$

for $g(x) = 2x^2 - x^3$ $\quad g'(x) = 4x - 3x^2 = x(4-3x)$ check $x = 0, \frac{4}{3}$

$x = 0$ means $y = 0$ at $(0,0)$ have minimum, but at

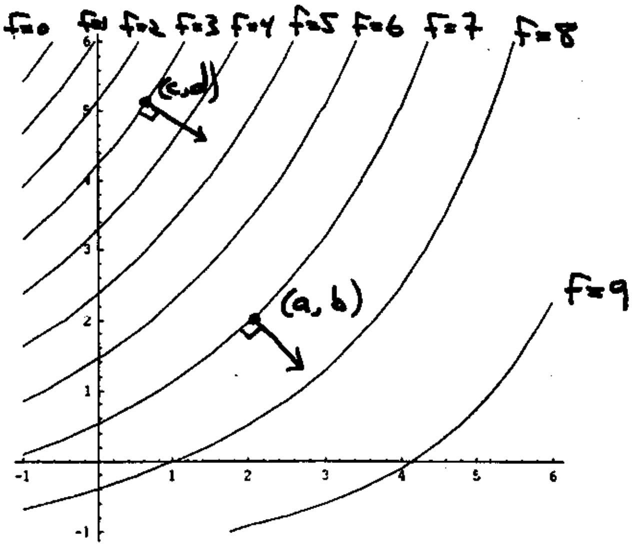
$x = \frac{4}{3}$ and so $y = -x = -\frac{4}{3}$ have $F\left(\frac{4}{3}, -\frac{4}{3}\right) = \frac{16}{9} + \frac{16}{9} - \frac{64}{27} = \frac{32}{9} - \frac{64}{27}$

$= \frac{96}{27} - \frac{64}{27} = \frac{32}{27} < 4$

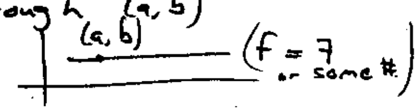
So max at $(0,-2)$ with $F(0,-2) = 4$

Question 3. (16 points total)

The contour plot of the function $f(x, y)$ is given in the figure below. For statements (i) to (iv) decide whether the statement is true or false, and give the reason. For part (v) draw your answers neatly on the figure below.



(i) (3 points) $f_x(a,b) \neq 0$ True, as if $f_x(a,b) = 0$ then this means the rate of change of f in the x-direction is zero, so the level curve through (a, b) would be parallel to x-axis (which it is not)



(ii) (3 points) $f_x(a,b) > f_x(c,d)$

False From the contours on the figure, the contours are closer together as one goes in the x direction from (c,d) compared to those from (a,b) so f(x,y) is steeper at (c,d), and $f_x(c,d) > f_x(a,b)$

(iii) (3 points) $f_x(a,b) < f_y(a,b)$

False $f_y(a,b)$ is in fact negative, as one can see from the contour plot → the contours go down in value as y increases from the point (a,b)

(iv) (3 points) $f_{xx}(a,b) > 0$ False, the contour lines are becoming more spaced out as one goes along the line $y = b$ through the point (a,b). This means that although $f_x(a,b)$ is positive that f_x is decreasing as x increases which means $f_{xx}(a,b)$ is negative (concave down)

(v) (4 points) Starting at (*a*, *b*) on the figure, draw a unit vector in the direction of $\nabla f(a,b)$. Then starting at the point (*c*, *d*), draw a unit vector in the direction of $\nabla f(c,d)$.

Question 4. (8 points total)

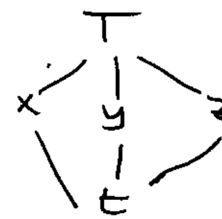
The moth is still flying. This time the moth's position at time t seconds is given by the position vector $\mathbf{r}(t) = \langle t\cos(\pi t),\ t\sin(\pi t),\ 10\rangle$ (for $0 \le t < \infty$)

Suppose the temperature at any point in space is given by $T(x,y,z) = 2xz + y^2 + z^3 + 40$.

What is the rate of change of the temperature per second as experienced by the moth at time $t = 1$?

T depends on x, y, z, and x, y, z depend on time t, so to figure out rate of change of temp. T per time (second) t, we need to compute $\frac{dT}{dt}$ → we can use the chain rule

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial T}{\partial z}\cdot\frac{dz}{dt}$$



$x = t\cos(\pi t)$ so $\frac{dx}{dt} = t(-\pi\sin(\pi t)) + \cos(\pi t)$

$y = t\sin(\pi t)$ so $\frac{dy}{dt} = t\,\pi\cos(\pi t) + \sin(\pi t)$

$z = 10$ so $\frac{dz}{dt} = 0$

so $$\frac{dT}{dt} = 2z\left(-\pi t\sin(\pi t) + \cos(\pi t)\right) + 2y\left(\pi t\cos(\pi t) + \sin(\pi t)\right) + \frac{\partial T}{\partial z}\cdot 0$$

at time $t = 1$ $x = -1$, $y = \sin(\pi) = 0$, $z = 10$,

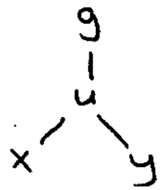
so $$\frac{dT}{dt} = 20\left(-\pi\sin(\pi) + \cos(\pi)\right) + 0 + 0 = -20$$

Question 5. (8 points total)

Let $F(x,y) = g(x^2 y)$ where g is a continuous function of one variable with continuous first and second derivatives. Calculate $F_{xy}(2,2) + F_{yx}(2,2)$ if you also know that $g'(8) = 3$ and $g''(8) = 1$.

So here we have $g(u)$ where $u(x,y) = x^2 y$

so $F_x = \dfrac{dg}{du} \cdot \dfrac{\partial u}{\partial x}$



or just $g'(u) \cdot 2xy$

$= g'(x^2 y) \cdot 2xy \longrightarrow$ looks like old single variable chain rule with y considered as a constant

then $F_{xy} = g'(x^2y) \cdot 2x + [g''(x^2 y) \cdot x^2] \cdot 2xy$

product rule

1st times deriv. of 2nd + deriv. of 1st times 2nd

so when $x = 2$, $y = 2$ and $g'(8) = 3$, $g''(8) = 1$

we get $F_{xy}(2,2) = 3 \cdot 4 + [1 \cdot 4] \cdot 8$

$= 12 + 32 = 44$

Now we have Clairault's Theorem which tells us that $F_{xy} = F_{yx}$ (as g has continuous 1st, 2nd derivatives) so $F_{yx}(2,2) = 44$ as well, so $F_{xy}(2,2) + F_{yx}(2,2) = 44 + 44 = 88$

(note $F_{yx} = g'(x^2 y) \cdot 2x + g''(x^2 y) \cdot x^2 \cdot 2xy$ as well)

Question 6. (14 points total)

Suppose $F(x, y)$ is a function such that $D_{\mathbf{u}}F(1,1) = 5$ and $D_{\mathbf{v}}F(1,1) = \sqrt{2}$, where $\mathbf{u} = \langle 1,0 \rangle$ and $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(a) (6 points) Find the vector $\nabla F(1,1)$

We know $D_u F(1,1) = \nabla F(1,1) \cdot \vec{u} = \nabla F(1,1) \cdot \langle 1,0 \rangle$

so if $\nabla F(1,1) = \langle a, b \rangle$, then $\langle a,b \rangle \cdot \langle 1,0 \rangle = 5$

so $a = 5$.

Next $D_v F(1,1) = \sqrt{2} = \nabla F(1,1) \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

so $\nabla F(1,1) = \langle 5, b \rangle$ implies $\frac{5}{\sqrt{2}} + \frac{b}{\sqrt{2}} = \sqrt{2}$

or $5 + b = 2$, so $b = -3$, so $\nabla F(1,1) = \langle 5, -3 \rangle$

(b) (4 points) What is the maximum possible value of $D_{\mathbf{w}}F(1,1)$ for any unit vector $\mathbf{w}$?

Note $D_w F(1,1) = \nabla F(1,1) \cdot \vec{w} = |\nabla F(1,1)|\,|\vec{w}| \cos\theta$

θ angle btw. $\vec{w}$ and $\nabla F(1,1)$

so if $\vec{w}$ points in $\nabla F(1,1) = \langle 5, -3 \rangle$ direction

then $D_w F(1,1) = |\nabla F(1,1)| = |\langle 5,-3 \rangle| = \sqrt{34}$

$\longrightarrow$ maximum possible value

(Also note gradient points in direction of max'l increase, so $D_w F(1,1)$ hits max'l value $|\nabla F(1,1)| = \sqrt{34}$)

(c) (4 points) Find a unit vector $\mathbf{w}$ such that $D_{\mathbf{w}}F(1,1) = 0$

$$D_w F(1,1) = \nabla F(1,1) \cdot \vec{w}$$
$$= \langle 5, -3 \rangle \cdot \vec{w}$$

$$\langle 5, -3 \rangle \cdot \langle 3, 5 \rangle = 0$$

(or $\langle 5, -3 \rangle \cdot \langle a, b \rangle = 0$ implies $5a - 3b = 0$, so if $a = 1$, $b = \frac{5}{3}$ $\Rightarrow \langle 1, 5/3 \rangle$)

but need unit vector $\vec{w}$,

so take $\vec{w} = \dfrac{\langle 3, 5 \rangle}{|\langle 3,5 \rangle|} = \left\langle \dfrac{3}{\sqrt{34}}, \dfrac{5}{\sqrt{34}} \right\rangle$

Question 7. (14 points total)

Captain Kurt of the starship Interprize has steered his starship towards a wormhole in space. The Interprize's current location is (0, 2, 2). The wormhole is quite oddly shaped, and its surface is given by the equation $-x^2+2y^2+2z^2=1$. Captain Kurt is really eager to get to the wormhole as soon as possible. Find a point on the wormhole's surface that is nearest to the Interprize so that Kurt knows where to send the ship. Be sure to write up your solution to this problem carefully, explaining all the steps involved in getting to your answer.

So we need to find a point (x,y,z) such that $g(x,y,z)=-x^2+2y^2+2z^2=1$ (i.e. on wormhole), that minimizes distance to $(0,2,2)$.

This distance is $\sqrt{(x-0)^2+(y-2)^2+(z-2)^2}$

nasty to have the $\sqrt{\ }$ so note that minimizing $(x-0)^2+(y-2)^2+(z-2)^2$ is equivalent

so minimize $f(x,y,z)=x^2+y^2-4y+4+z^2-4z+4$

... Lagrange multipliers $\nabla f=\lambda\nabla g$

$\nabla f=\langle 2x, 2y-4, 2z-4\rangle=\lambda\langle -2x, 4y, 4z\rangle$

so ① $2x=-\lambda 2x$ and ④ $g(x,y,z)=1$

② $2y-4=4\lambda y$ From ① either $x=0$, or $\lambda=-1$

③ $2z-4=4\lambda z$

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if $x=0$ then y and z can't both be 0, so from ② $\frac{2y-4}{4y}=\lambda$ and from ③ $\frac{2z-4}{4z}=\lambda$

so $\frac{1}{2}-\frac{1}{y}=\lambda=\frac{1}{2}-\frac{1}{z}$ implies $\frac{1}{y}=\frac{1}{z}$ or $y=z$

from ④ then $-0^2+2y^2+2y^2=1 \Rightarrow y=\pm\frac{1}{2}=z$,

so get points $(0,\frac{1}{2},\frac{1}{2})$ and $(0,-\frac{1}{2},-\frac{1}{2})$

clearly $(0,\frac{1}{2},\frac{1}{2})$ is closer to $(0,2,2)$

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if $\lambda=-1$ then from ② $2y-4=-4y \Rightarrow 6y=4$, $y=\frac{2}{3}$

and same from ③ $\Rightarrow z=\frac{2}{3}$.

Then from ④ $-x^2+2(\frac{2}{3})^2+2(\frac{2}{3})^2=1$ implies $-x^2=1-4\cdot\frac{4}{9}$

so $x=\pm\frac{\sqrt{7}}{3}$

so get points $(\pm\frac{\sqrt{7}}{3},\frac{2}{3},\frac{2}{3})$

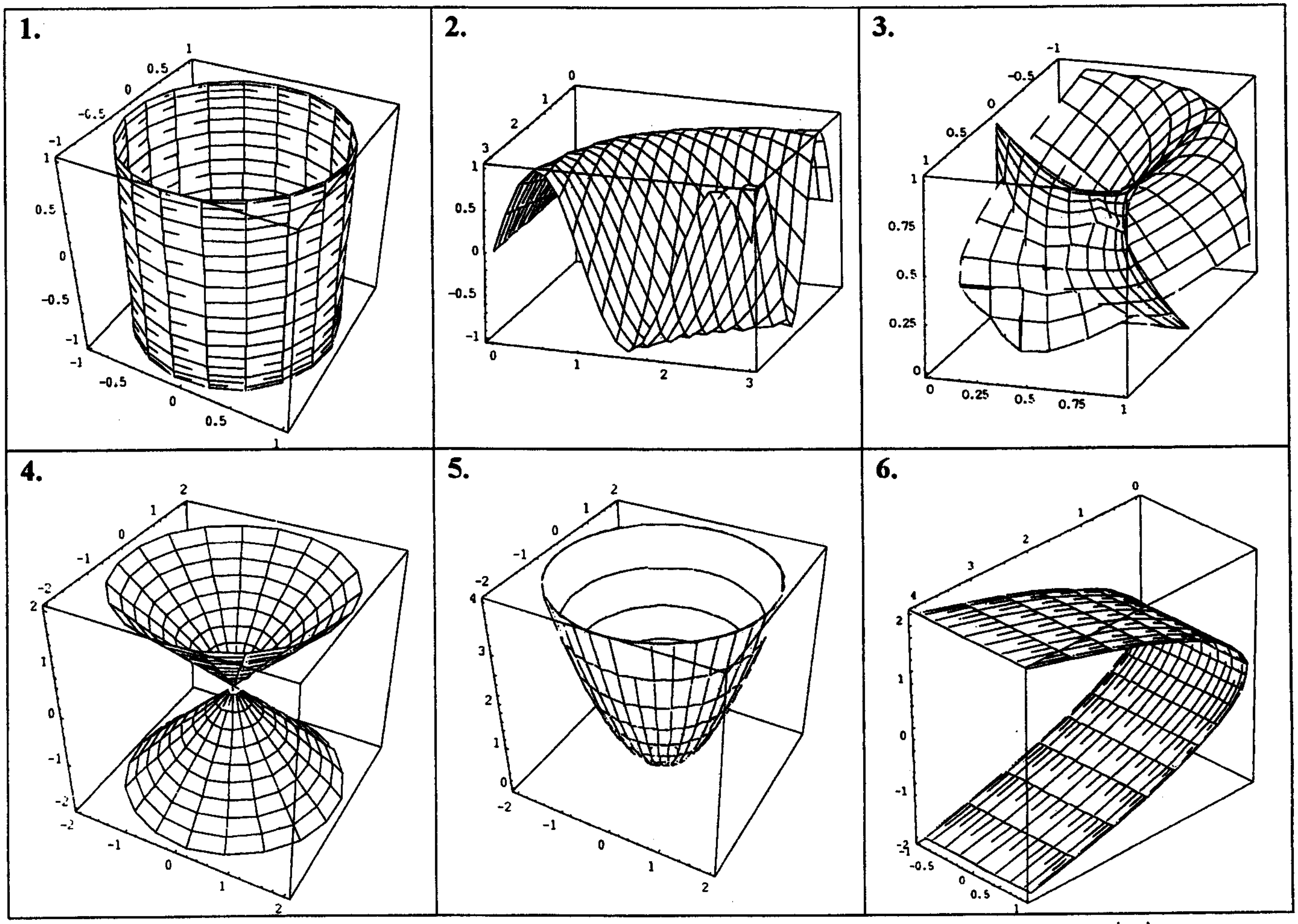
Check distances! from $(0,\frac{1}{2},\frac{1}{2})$ distance to $(0,2,2)$ is $(\frac{3}{2})^2+(\frac{3}{2})^2=\frac{18}{4}$

from $(\pm\frac{\sqrt{7}}{3},\frac{2}{3},\frac{2}{3})$ to $(0,2,2)$ distance is $\frac{7}{9}+\frac{16}{9}+\frac{16}{9}=\frac{39}{9}$

Note $\frac{39}{9}=4\frac{1}{3}$ and $\frac{18}{4}=4\frac{1}{2}$, so either point $(\pm\frac{\sqrt{7}}{3},\frac{2}{3},\frac{2}{3})$ is closest

Question 8. (12 points total, 2 points each)

Welcome to Sin City! Match the parametric surfaces with their parametrization. No justification is needed. Write the number of the appropriate surface in the blank next to the correct parametrization.



$z = x^2 + y^2$ paraboloid

cylinder: circular cross-sections $\underbrace{\text{same radius}}$

sinusoidal cross sections in x, and y directions

5 $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^2 \rangle$	1 $\mathbf{r}(u,v) = \langle \cos v, \sin v, \sin u \rangle$ runs −1 to 1	2 $\mathbf{r}(u,v) = \langle u, v, \sin(uv) \rangle$
4 $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u \rangle$	6 $\mathbf{r}(u,v) = \langle u^2, \sin(v), u \rangle$	3 $\mathbf{r}(u,v) = \langle \sin(uv), \sin v, \sin u \rangle$

$z = \pm\sqrt{x^2+y^2}$ cone

parabola for $x = z^2$

yuck!, all that is left is 3