Name: Answers

Math 21a Midterm 2 Tuesday, April 16th, 2002

Please circle your section:

Melissa Liu Roger Hong (CA) MWF 9-10	Christophe Cornut David Shim (CA) MWF 10-11	Spiro Karigiannis Michael Simonetti (CA) MWF 10-11
Spiro Karigiannis Gloria Hou (CA) MWF 11-12	Andy Engelward Nathan Moore (CA) T/Th 10-11:30	Andy Engelward Alexey Gorshkov (CA) T/Th 11:30-12

Question	Points	Score
1	10	
2	20	
3	20	
4	18	
5	12	
6	20	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order - you should move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers will receive little or no credit! Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one 3 by 5 inch file card with formulas on it during the test, but you are not allowed to use any other notes, or calculators during this test.

Good luck! Focus and do well!

Question 1. (10 points total)

Suppose the function $f(x,y)=2x+3x^2y-y^3$ gives the temperature in degrees Farenheit at the point (x, y) on a tin roof of dimensions 100 feet by 50 feet (i.e. $0 \le x \le 100$ and $0 \le y \le 50$ and x and y are measured in feet). Suppose a cat is walking around on the tin roof, and her position is given by the position vector $\vec{r}(t)=\langle e^t(2+\cos(t)), \ln(e+t)\rangle$ at time t seconds.

What is the rate of change of the temperature per second as experienced by the cat at time $t=0$?

rate of change of temperature per second is $\frac{df}{dt}$,

we know f in terms of x, y, and

x, y are given in terms of t $(\vec{r}(t)=\langle e^t(2+\cos t), \ln(e+t)\rangle$
($x(t)$ ↑, $y(t)$ ↑)

so we need to use chain rule to find $\frac{df}{dt}$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2+6xy)\cdot\left(e^t(-\sin t)+e^t(2+\cos t)\right) + (3x^2-3y^2)\left(\frac{1}{e+t}\right)$$

Now when $t=0$ $\vec{r}(0)=\langle e^0(2+\cos(0)), \ln(e+0)\rangle$
$=\langle 3, 1\rangle$

so $x(0)=3$ $y(0)=1$

and
$$\left.\frac{df}{dt}\right|_{t=0} = (2+6\cdot 3\cdot 1)\left(e^0(-\sin(0))+e^0(2+\cos(0))\right) + (3\cdot 3^2-3\cdot 1^2)\left(\frac{1}{e+0}\right)$$

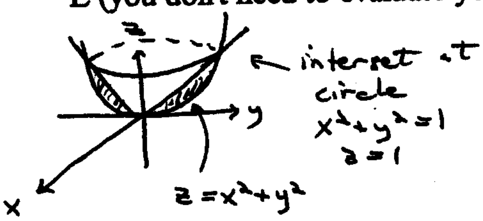
$$\boxed{=60+\frac{24}{e}}$$ degrees/sec.

Note – this question is asking for $\frac{df}{dt}$, <u>not</u> a directional derivative, which leads to a different result

Question 2 (20 points total)

Let E be the solid region bounded by the cone $x^2+y^2=z^2$ and the paraboloid $x^2+y^2=z$.

(a) (5 points) Set up a triple integral using rectangular (cartesian) coordinates that gives the volume of E (you don't need to evaluate your integral!).



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dy\,dx$$

(or switch dx, dy and limits $\sqrt{1-y^2}$)

(b) (5 points) Set up a triple integral using cylindrical coordinates that gives the volume of E (again, just set up the integral, don't evaluate it).

in cylindrical coords. surfaces are $r^2=z^2$, $r^2=z$ i.e. $r^2 \le z \le r$

or $r=z$ — upper half

over circle region $x^2+y^2 \le 1$

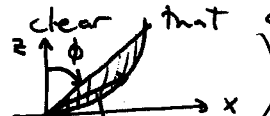
or $0 \le r \le 1$

$0 \le \theta \le 2\pi$

so... $\displaystyle\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$

(c) (5 points) Set up a triple integral using spherical coordinates that gives the volume of E (again, just set up the integral, don't evaluate it).

Although it's clear that $0 \le \theta \le 2\pi$ and $\pi/4 \le \phi \le \pi/2$ (clear from cross section:



), finding limits for ρ is a bit more involved: the maximum for ρ is $\sqrt{2}$ (distance to (1,0,1) for example), which ρ is bounded by the surface of the paraboloid as shown, which in spherical coord.s is... $x^2+y^2=z \Rightarrow (\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2=\rho\cos\phi$

so $\rho^2\underbrace{\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta}_{=1}\sin^2\phi=\rho^2\sin^2\phi=\rho\cos\phi$, or $\rho=\dfrac{\cos\phi}{\sin^2\phi}$ — so total volume integral is

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\frac{\cos\phi}{\sin^2\phi}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(d) (5 points) Now, evaluate the triple integral $\iiint_E (x^2z+y^2z)\,dV$ using any coordinate system you want. Please be sure to simplify your answer as much as possible.

Cylindrical is the way to go → simplest integration as $r^2=x^2+y^2$, so integrand $x^2z+y^2z=r^2z$, and so in cylindrical coord.s you end with an integral involving just powers of r and z:

(picking the right integral to do was key to doing this problem!)

$$=\int_0^{2\pi}\int_0^1\int_{r^2}^{r}(r^2z)\cdot r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^3z\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left(\frac{r^3z^2}{2}\Big|_{z=r^2}^{z=r}\right)dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1\left(\frac{r^5}{2}-\frac{r^7}{2}\right)dr\,d\theta=\int_0^{2\pi}\left(\frac{r^6}{12}-\frac{r^8}{16}\right)\Big|_{r=0}^{r=1}d\theta=2\pi\left(\frac{1}{12}-\frac{1}{16}\right)$$

$$=2\pi\left(\frac{1}{48}\right)=\boxed{\frac{\pi}{24}}$$

Question 3. (20 points total)

Suppose a cylindrical storage tank of radius R (in feet) is being built. The tank can be modeled in xyz coordinates as a cylinder with base z = 0, cylindrical side given by the usual equation $x^2 + y^2 = R^2$ (where x, y and z are all measured in feet). The roof of the tank is being built at a slant, with equation $z = 30 + 2x + y$, so that the rain runs off (thus, for instance, right over the center of the tank, where $x = y = 0$, the height of the roof is 30 feet).

(a) (8 points) What would the radius of the tank have to be (i.e. the value of R), so that the area of the roof comes out to be exactly 200 square feet?

We know surface area $= \iint_D |\vec{F}_u \times \vec{F}_v| \, dA$, but need to have a parametrization of roof. In this case, since it's just the graph of the function $z = 30 + 2x + y$, then the easiest parametrization is just $\langle x, y, 30+2x+y \rangle$ (with $x^2+y^2 \le R^2$). Then $\vec{F}_x = \langle 1, 0, 2 \rangle$ $\vec{F}_y = \langle 0, 1, 1 \rangle$, and $\vec{F}_x \times \vec{F}_y = \langle -2, -1, 1 \rangle$ so $|\vec{F}_x \times \vec{F}_y| = \sqrt{6}$ (or could just know alternative surface area formula in which $|\vec{F}_x \times \vec{F}_y| = \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{6}$)

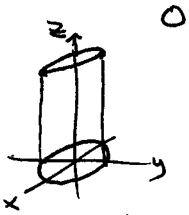
Now find $\iint_D \sqrt{6}\, dA$ where D is the $x^2+y^2 \le R^2$ circular base of the storage tank. Aha, polar coordinates are key here...

$\iint_D \sqrt{6}\, dA = \int_0^{2\pi} \int_0^R \sqrt{6}\, r\,dr\,d\theta = \int_0^{2\pi} \left(\sqrt{6} \frac{r^2}{2} \Big|_{r=0}^{r=R} \right) d\theta = \pi R^2 \sqrt{6}$ (or note $\iint_D dA = \pi R^2$ since it's just the area of base circle!)

Now $\pi R^2 \sqrt{6} = 200$

So $R = \sqrt{\dfrac{200}{\pi\sqrt{6}}}$ feet

(b) (6 points) Write down a triple integral and a double integral both of which give the volume of the tank (use the radius you found in part a, or if you didn't solve that part, just write in the radius as "R"!)



$0 \le z \le 30 + 2x + y$ as $x^2 + y^2 \le R^2$ so either have

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_0^{30+2x+y} dz\,dy\,dx \quad \text{or} \quad \int_0^{2\pi} \int_0^R \int_0^{30+2r\cos\theta + r\sin\theta} r\,dz\,dr\,d\theta$$

as a double ... either $\displaystyle\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (30+2x+y)\,dy\,dx$ or $\displaystyle\int_0^{2\pi} \int_0^R (30 + 2r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$

(c) (6 points) Now evaluate either one of the integrals your wrote down in part (b)

Note – could either have "R" or answer to part (a) in for R

Cylindrical or polar is much the better choice (note – the tank is cylindrical → that should be a bit of a tip off as to which way to go here!)

Double integral $\displaystyle\int_0^{2\pi} \int_0^R (30 + 2r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$

$$= \int_0^{2\pi} \left(\frac{30r^2}{2} + \frac{2r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta \right) \Big|_{r=0}^{r=R} d\theta = \int_0^{2\pi} \left(15R^2 + \tfrac{2}{3}R^3\cos\theta + \tfrac{1}{3}R^3\sin\theta \right) d\theta$$

$$= \left(15R^2\theta + \tfrac{2}{3}R^3\sin\theta - \tfrac{1}{3}R^3\cos\theta \right) \Big|_{\theta=0}^{2\pi} = 15R^2\, 2\pi = 30\pi R^2$$

with $R = \sqrt{\frac{200}{\pi\sqrt{6}}}$ this equals $\frac{6000}{\sqrt{6}}$

or $1000\sqrt{6}$ cubic feet.

Question 4. (18 points total)

A small, hitherto undiscovered, moon of Jupiter has radius 100 miles. Suppose we model this moon as a solid sphere with the z-axis runs along the moon's axis of rotation (i.e. the z-axis goes through the moon's "north" and "south" poles), the origin is at the center of the moon, and its equator lies in the xy-plane. Thus the surface of the moon is given by the equation $x^2 + y^2 + z^2 = 100^2$.

So far science has discovered that the temperature function, $T(x, y, z)$, measured in Farenheit, for points on the surface of the moon is $T(x, y, z) = 3x + kz$, where *k* is an unknown constant (note $T(x, y, z)$ is a three variable function that happens to be independent of *y*). It's also known that the hottest point on this moon is at the point with (x, y, z) coordinates equal to (60, 0, -80).

Using the Lagrange multiplier method, it is possible to determine the value of the constant *k*.

(a) (6 points) Find the value of *k*, and also the temperature at the hottest point.

Pretend we're solving with Lagrange method: $\nabla T = \langle 3, 0, k \rangle = \lambda \nabla g$ where $g(x,y,z) = x^2+y^2+z^2 = 100^2$, so $\lambda \nabla g = \lambda \langle 2x, 2y, 2z \rangle$

so $3 = \lambda 2x$, $0 = \lambda 2y$, $k = \lambda 2z$ so $\lambda = \frac{3}{2x}$ ... $\lambda = \frac{k}{2z}$

Aha, $\frac{3}{2x} = \frac{k}{2z}$, but we know $x = 60$ and $z = -80$ at the hottest point, so $\frac{3}{2 \cdot 60} = \frac{k}{2 \cdot (-80)} \Rightarrow \boxed{k = -4}$

→ and then $T(60, 0, -80) = 3 \cdot 60 - 4(-80) = \boxed{500°}$

(b) (12 points) What are the coordinates of the point on the surface where the lowest temperature occurs? Use the value of *k* that you found in part (a) in the temperature function, or if you weren't able to solve part (a) use another temperature function $T(x, y, z) = 8x + 6z$

if $T(x,y,z) = 3x - 4z$ then follow standard Lagrange method:

$g(x,y,z) = x^2+y^2+z^2 = 100^2$, $\nabla T = \langle 3, 0, -4 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y, 2z \rangle$

so $3 = 2\lambda x$, $0 = 2\lambda y$, $-4 = 2\lambda z$

→ either $\lambda = 0$ or $y = 0$, but λ can't equal 0 because $3 = 2\lambda x$, so $y = 0$,

then, as in part (a) $\lambda = \frac{3}{2x}$ and $\lambda = \frac{-4}{2z}$,

so $\frac{3}{2x} = \frac{-4}{2z} \Rightarrow 3z = -4x$, or $z = -\frac{4}{3}x$

So using 4th equation $x^2+y^2+z^2 = 100^2$, $x^2 + 0^2 + \left(-\frac{4}{3}x\right)^2 = 100^2$

so $x^2\left(1 + \frac{16}{9}\right) = 100^2$ $x^2\left(\frac{25}{9}\right) = 100^2$, $x = \pm\sqrt{\frac{9}{25}}\, 100$

$= \pm \frac{3}{5} \cdot 100 = \pm 60$

So two points to check are

$(60, 0, -\frac{4}{3} \cdot 60) = (60, 0, -80)$ and $(-60, 0, 80)$

Temps: $T(60, 0, -80) = 500$ $T(-60, 0, 80) = -500$

so lowest temp. occurs at $\boxed{(-60, 0, 80)}$

Alternate answer → if you used $T(x,y,z) = 8x + 6z$, same approach, but end up with $x^2\left(1 + \frac{9}{16}\right) = 100^2 \Rightarrow x = \pm 80$, $z = \pm 60$, check both points - coldest at $\boxed{(-80, 0, -60)}$ → temperature is -1000

Question 5. (12 points total)

Evaluate the double integral $\int_0^1 \int_{x^2}^{x} \frac{2x}{y} e^y \, dy dx$. Be sure to simplify your answer completely.

Try outright ... $= \int_0^1 \int_{x^2}^{x} \frac{2x}{y} e^y \, dy \, dx$, by parts ...

$u = \frac{2x}{y} \qquad dv = e^y \, dy$

$du = -\frac{2x}{y^2} dy \qquad v = e^y$ ... gets uglier

or $u = e^y \qquad dv = \frac{2x}{y} dy$

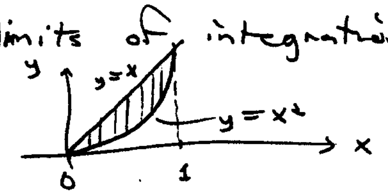
$du = e^y dy \qquad v = 2x \ln|y|$ ... still gets uglier.

Hmm, recall that with double integrals we have a choice of order of integration...

region is described in the limits of integration

as $x^2 \leq y \leq x$

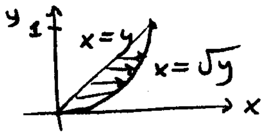
as $0 \leq x \leq 1$ ...



reversing order we would get

$y \leq x \leq \sqrt{y}$

as $0 \leq y \leq 1$



So integral becomes $\int_0^1 \int_y^{\sqrt{y}} \frac{2x}{y} e^y \, dx \, dy$

Try it ... $= \int_0^1 \left(\frac{x^2}{y} e^y \Big|_{x=y}^{x=\sqrt{y}} \right) dy = \int_0^1 \left(\frac{y}{y} e^y - \frac{y^2}{y} e^y \right) dy$

$= \int_0^1 (e^y - ye^y) dy = \int_0^1 e^y dy - \int_0^1 ye^y dy$

Now $\int ye^y dy$ is an easy by-parts integral: $u = y \quad dv = e^y dy$, $du = dy \quad v = e^y$

$\int ye^y dy = ye^y - \int e^y dy = ye^y - e^y$ (+c!)

So integral equals $\int_0^1 e^y dy - \left[ye^y - e^y \right]_{y=0}^{y=1} = \left(e^y - ye^y + e^y \right) \Big|_0^1$

$= (e - e + e) - (1 - 0 + 1)$

$\boxed{I = e - 2}$

Question 6. (20 points total)

You are in the dessert. It is very hot. You are thirsty. According to your map, there should be an oasis 5 miles away, in the northeast direction. Locally the temperature is given by the function

$$T(x,y) = -2x^3 + 3x^2 + 54x + 6xy + 3y^2 + 120$$

where x and y measure the distance in miles away from you in the east and north directions respectively, so that you are currently at the point (0,0). Someone has told you that you are less than 5 miles away from the coolest point in the dessert.

(a) (5 points) In what direction should you go to experience the fastest immediate *decrease* in temperature?

gradient points in direction of fastest increase, so opposite direction is fastest decrease: $\nabla T = \langle -6x^2 + 6x + 54 + 6y, 6x + 6y \rangle$

at (0,0) $\nabla T(0,0) = \langle 54, 0 \rangle$, so opposite direction is in direction of $\langle -54, 0 \rangle$, which is **due west**

(b) (10 points) In which direction should you go to get to the coldest point in the dessert (assuming that the coldest point is indeed less than 5 miles away, and that it's at a local minimum of $T(x,y)$)?

So, find local minimum of $T(x,y)$ (within 5 miles)...

have $\nabla T = \langle -6x^2 + 6x + 54 + 6y, 6x + 6y \rangle$. Set $\nabla T = \langle 0, 0 \rangle$

so $6x + 6y = 0 \Rightarrow y = -x$

then $-6x^2 + 6x + 54 + 6y = -6x^2 + 6x + 54 - 6x = -6x^2 + 54 = 0$

so $x^2 = 9$, $x = \pm 3$

Two points to check, then:

$(+3, -3)$ and $(-3, +3)$. ($y = -x$) Check with 2nd derivative test:

"D" $= T_{xx} \cdot T_{yy} - (T_{xy})^2 = (-12x + 6)(6) - (6)^2$

at $(3, -3)$ $D = (-30) \cdot 6 - 6^2 < 0 \Rightarrow$ saddlepoint,

at $(-3, 3)$ $D = (36 + 6) \cdot 6 - 6^2 > 0$ and $T_{xx} = 42 > 0$ so local min.

Aha, go towards $(-3, 3)$ which is **North west**

(c) (5 points) You are really thirsty and so you decide to go directly to the oasis, which is 5 miles away to the northeast. What is the rate of change of the temperature in degrees per mile right as you start walking directly towards the oasis?

Finally, a directional derivative question!

oasis is in the northeast direction or $(1, 1)$

direction, $\vec{u}$, unit vector in same direction is then $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

We already know $\nabla T(0,0) = \langle 54, 0 \rangle$, so rate of change of temperature in direction of oasis $D_{\vec{u}} T = \nabla T \cdot \vec{u} = \frac{54}{\sqrt{2}} = $ **$27\sqrt{2}$ deg/mile**