Math 2la PDE Handout

Suggested Problems - Answers Spring 2002

(1) (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x) + (-6x) = 0$ so it is a solution

(b) $(12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$ yes, it's a solution

(c) $(-2\sin(x-y)e^{(x-y)}) + (-2\sin(x-y)e^{(x-y)}) \neq 0$ not a solution

(d) $\left(\frac{2xy}{(x^2+y^2)^2}\right) + \left(\frac{-2xy}{(x^2+y^2)^2}\right) = 0$ yes it's a solution

(e) $(3998y) + (4004x) \neq 0$ so no, not a solution

(f) $(e^x(\sin y + 2\cos y)) + (e^x(-\sin y - 2\cos y)) = 0$, it is a solution

(2) (a) $\frac{\partial^2 u}{\partial t^2} = -4(\sin(x+2t) + \cos(x-2t)) = 4\frac{\partial^2 u}{\partial x^2}$, yes it's a solution with $c^2 = 4$
(Then assuming c positive, $c = 2$)

(b) $\frac{\partial^2 u}{\partial t^2} = -\frac{1}{t^2} \neq c\frac{\partial^2 u}{\partial x^2}$ for any constant c, so it's not a solution

(c) $\frac{\partial^2 u}{\partial t^2} = 24x = \frac{\partial^2 u}{\partial x^2}$ so, yes, it's a solution with $c^2 = 1$
$\therefore c = 1$

(d) $\frac{\partial^2 u}{\partial t^2} = -100^2 \sin(100x)\sin(100t) = \frac{\partial^2 u}{\partial x^2}$, is a solution, $c = 1$

(e) $\frac{\partial^2 u}{\partial t^2} = 4004x \neq c\frac{\partial^2 u}{\partial x^2}$, so not a solution

(f) $\frac{\partial^2 u}{\partial t^2} = 8 = 4\frac{\partial^2 u}{\partial x^2}$, so yes, it's a solution with $c^2 = 4$
$(c = 2)$

(3) (a) here the only derivative is with respect to x, so we can treat this as a simple ordinary differential equation, and treat y as just an extra parameter - ending up multiplying by an arbitrary function $g(y)$...

so if we saw $\frac{d^2 f}{dx^2} + 9f = 0$ (where f is a fct. just of x)

Then the solution would be $A\cos(3x) + B\sin(3x)$
(from Math 1b section on differential equations → this is the general solution to an undamped spring equation)

(3) (a) continued.. then the general solution for

$\frac{\partial^2 u}{\partial x^2} + 9u = 0$ is just $(A\cos(3x) + B\sin(3x))\, g(y)$

where A, B are arbitrary constants, and $g(y)$ is a function of y alone

(b) ~~This problem is extremely difficult as stated. My apologies for not having stated it correctly as just~~ $\cancel{\frac{\partial u}{\partial y} + 2yu = 0}$ ~~(not "~~ $\cancel{\frac{\partial^2 u}{\partial y^2}}$ ~~" !~~ (already corrected in original)

In the correct form one solves this as a simple ODE in the variable y: solution ce^{-y^2}

Then multiply this by an arbitrary function $g(x)$, of x alone, so general solution is $g(x)e^{-y^2}$

(c) $\frac{\partial u}{\partial x} = 2xyu$, again solve as an ODE in x alone, then the solution is determined up to a multiple by a function in y. Again, basic Math 1b technique solves the ODE: divide by u, "multiply" by dx...

get $\int \frac{1}{u}\, du = \int 2xy\, dx$, so $\ln|u| = x^2 y + c_1$

then as an ODE: $u = \pm e^{x^2 y + c_1} = ce^{x^2 y}$

so with u a function of x and y $u(x,y) = g(y)e^{x^2 y}$

(d) $\frac{\partial^2 u}{\partial x \partial y} = 0$ just integrate with respect to x:

get $\frac{\partial u}{\partial y} = f(y)$ (since then $\frac{\partial}{\partial x}(f(y)) = 0$

so $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 0$

Then integrate with respect to y: get $\int f(y)\, dy = g(y) + h(x)$ (where $\frac{d}{dy}(g(y)) = f(y)$)

so general solution is just $u(x,y) = g(y) + h(x)$, where g and h are arbitrary functions of y and x alone, respectively

(4) (a) since $\frac{\partial^2 u}{\partial x^2} = 0$, then from the perspective of the variable x, $u(x,y) = ax + b$,

where a and b could be functions of y, i.e. $u(x,y) = a(y)x + b(y)$. Next, since $\frac{\partial^2 u}{\partial y^2} = 0$, then $a(y)$ and $b(y)$ can at most be linear functions in y, i.e $a(y) = cy + d$, $b(y) = fy + g$, so...

(4)(a) the general solution is $u(x,y) = (cy+d)x + fy + g$
$= cxy + dx + fy + g$,
where c, d, f, g are arbitrary constants.

(b) more straightforward than (4)(a):
since $\frac{\partial u}{\partial x} = 0$, then $u(x,y) = f(y)$, where f is a function of y alone. but $\frac{\partial u}{\partial y} = 0$ so $f(y) = k$, a constant,
so $u(x,y) = k$ some constant

(c) since $\frac{\partial^2 u}{\partial x^2} = 0$, then $u(x,y) = a(y)x + b(y)$
again, as in (4)(a) then $\frac{\partial^2}{\partial x \partial y}\left(a(y)x + b(y)\right) = a'(y)$
so $a'(y) = 0$ which implies $a(y) = k$, in which case $u(x,y) = kx + b(y)$, k a constant, b a function of y

(d) since $\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial^2 u}{\partial y^2}$, then we already know from (4)(a) that the general solution to this set of PDE's is $cxy + dx + fy + g$. Now add in the extra condition $\frac{\partial^2 u}{\partial y \partial x} = 0$: $\frac{\partial^2}{\partial y \partial x}(cxy + dx + fy + g) = c$
so $c = 0$, and $u(x,y) = dx + fy + g$, d, f, g constants

(5) We start by knowing the general solution to the wave equation is $\left(A\cos\left(\frac{cn\pi}{l}t\right) + B\sin\left(\frac{cn\pi}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)$

(Note - on the final you will be given this solution if you need to use it - you don't need to memorize this!!)

Now length $l = \pi$, and $c = 1$ (we'll assume c is positive) so in these four questions we'll be working with the general solution
$$u(x,t) = \left(A\cos(nt) + B\sin(nt)\right)\sin(nx)$$
where A, B are constants, n an integer

(a) We need $u(x,0) = 0$ and $u_t(x,0) = \sin(4x)$,
so start with $u(x,0) = \left(A\cos(n\cdot 0) + B\sin(n\cdot 0)\right)\sin(nx)$
$= (A + B\cdot 0)\sin(nx) = A\sin(nx)$
since $u(x,0) = 0$, then $A = 0$

(5) (a) continued so $u(x,t) = B\sin(nt)\sin(nx)$.
Next $u_t(x,0) = nB\cos(n\cdot 0)\sin(nx) = nB\sin(nx)$
so $u_t(x,0) = \sin(4x)$ means $n=4$, $B=\frac{1}{4}$,
so $u(x,t) = \frac{1}{4}\sin(4t)\sin(4x)$

(b) here everything is the same as in (a) up to
$u_t(x,0) = nB\sin(nx)$ now equals $-0.1\sin(2x)$
so $n=2$, $B=-0.05$, $u(x,t) = -0.05\sin(2t)\sin(2x)$

(c) now $u(x,0) = A\sin(nx)$ is supposed to equal $0.1\sin(x)$
so $A=0.1$, $n=1$, and $u(x,t) = (0.1\cos(t) + B\sin(t))\sin(x)$
next $u_t(x,0) = (-0.1\sin(0) + B\cos(0))\sin(x) = B\sin x$
so since it's supposed to equal $-0.2\sin(x)$,
then $B=-0.2$, and $u(x,t) = (0.1\cos(t) - 0.2\sin(t))\sin(x)$

(d) now $u(x,0) = A\sin(nx) = 0.3\sin(6x)$ so $A=0.3$, $n=6$
next $u_t(x,0) = (-0.3\sin(0) + 6B\cos(0))\sin(6x) = 6B\sin(6x)$
so $6B = -0.2$, and $u(x,t) = \left(0.3\cos(6t) - \frac{1}{30}\sin(6t)\right)\sin(6x)$

(6) To use the separation of variables technique we assume that $u(x,y) = F(x)G(y)$, so
$$\frac{\partial u}{\partial x} = F'(x)G(y) \quad \text{and} \quad \frac{\partial u}{\partial y} = F(x)G'(y)$$

(a) so $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = F'(x)G(y) - F(x)G'(y) = 0$,
so $F'(x)G(y) = F(x)G'(y)$, or $\dfrac{F'(x)}{F(x)} = \dfrac{G'(y)}{G(y)}$
we know that this must equal a constant, k, (check the text if you are confused by this!)
so $F'(x) = kF(x)$ and $G'(y) = kG(y)$
then $F(x) = c_1e^{kx}$ $G(y) = c_2e^{ky}$, (both by Math 1b, ODEs)
so $u(x,y) = F(x)G(y) = Ce^{kx+ky} = Ce^{k(x+y)}$
C, k arbitrary constants

(b) so $F'(x)G(y) - yF(x)G'(y)$, so $\dfrac{F'(x)}{F(x)} = \dfrac{yG'(y)}{G(y)} = k$

(6) (b) continued, so $F'(x) = kF(x)$, $yG'(y) = kG(y)$
so $F(x) = C_1 e^{kx}$, and $G(y) = C_2 y^k$
(Math 1b) (trial and error - if this were on the final, you would probably be given this solution - don't worry!)

so $u(x,y) = C(e^{kx})y^k$

(c) so $xF'(x)G(y) - yF(x)G'(y) = 0$, so $\frac{xF'(x)}{F(x)} = \frac{yG'(y)}{G(y)} = k$

same as in (b), then $F(x) = C_1 x^k$, $G(y) = C_2 y^k$
so $u(x,y) = C(xy)^k$

(d) now $yF'(x)G(y) - xF(x)G'(y) = 0$, so $\frac{F'(x)}{xF(x)} = \frac{G'(y)}{yG(y)} = k$

so $F'(x) = kxF(x)$, $G'(y) = kyG(y)$
solutions: $\int \frac{1}{F}\,dF = \int kx\,dx \ \ldots \ \ln|F| = \frac{k}{2}x^2 + c$, $F(x) = C_1 e^{\frac{k}{2}x^2}$
similarly $G(y) = C_2 e^{\frac{k}{2}y^2}$, so $u(x,y) = Ce^{\frac{k}{2}(x^2+y^2)}$

(e) here $F'(x)G(y) + F(x)G'(y) = 2(x+y)F(x)G(y) = 2xF(x)G(y) + 2yF(x)G(y)$
then splitting into x, y: $(F'(x) - 2xF(x))G(y) + F(x)(G'(y) - 2yG(y)) = 0$

so $\frac{F'(x) - 2xF(x)}{F(x)} = \frac{-G'(y) + 2yG(y)}{G(y)} = k$, and $F'(x) - 2xF(x) = kF(x)$
$G'(y) - 2yG(y) = -kG(y)$

so $F'(x) = F(x)(2x+k)$, $G'(y) = G(y)(2y-k)$
solutions: $\int \frac{1}{F}\,dF = \int (2x+k)\,dx$ leads to $F(x) = Ce^{x^2+kx}$
so $u(x,y) = Ce^{x^2+kx}e^{y^2-ky} = Ce^{x^2+y^2+k(x-y)}$

(f) so $\frac{\partial^2 u}{\partial x \partial y} = F'(x)G'(y) = u = F(x)G(y)$, so $\frac{F'(x)}{F(x)} = \frac{G(y)}{G'(y)} = k$
then $F'(x) = kF(x) \Rightarrow F(x) = C_1 e^{kx}$
and $G'(y) = \frac{1}{k}G(y) \Rightarrow G(y) = C_2 e^{\frac{1}{k}y}$

so $u(x,y) = Ce^{(kx + \frac{1}{k}y)}$