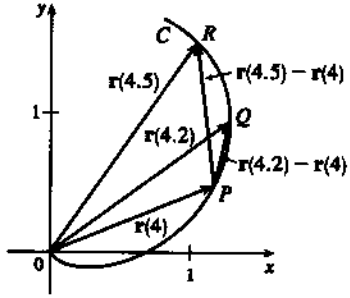
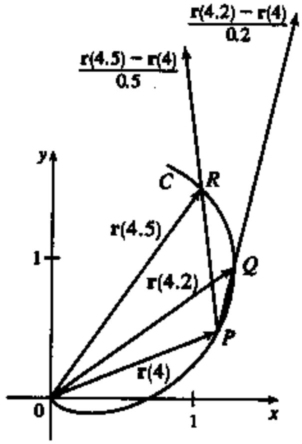
10.2 Derivatives and Integrals of Vector Functions

1. (a)



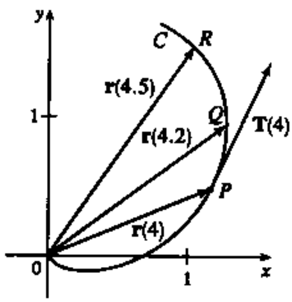
(b) $\dfrac{\mathbf{r}(4.5)-\mathbf{r}(4)}{0.5} = 2\,[\mathbf{r}(4.5)-\mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5)-\mathbf{r}(4)$. $\dfrac{\mathbf{r}(4.2)-\mathbf{r}(4)}{0.2} = 5\,[\mathbf{r}(4.2)-\mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2)-\mathbf{r}(4)$.



(c) By Definition 1, $\mathbf{r}'(4) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(4+h)-\mathbf{r}(4)}{h}$.

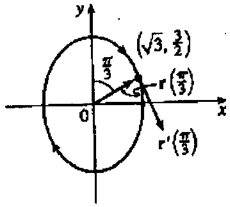
$\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.



6. $x = 2\sin t$, $y = 3\cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

(a), (c)



(b) $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

18. $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t}\rangle \Rightarrow \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \Rightarrow$

$\mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0\rangle = \langle 2, -2, 1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$. Then

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 2, -2, 1\rangle = \langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\rangle$. $\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \Rightarrow$

$\mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0\rangle = \langle 4, 4, 4\rangle$.

$$\mathbf{r}'(t)\cdot\mathbf{r}''(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \cdot \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle$$

$$= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t})$$

$$= 8e^{4t} - 8e^{-4t} + (8t^2 + 12t + 4)e^{4t} = (8t^2 + 12t + 12)e^{4t} - 8e^{-4t}$$

21. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, so $\mathbf{r}'(t) = \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t})\rangle = \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle$. The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0\rangle = \langle -1, 1, -1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1\rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1\cdot t = t$, $z = 1 + (-1)t = 1 - t$.

28. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle θ of intersection, we proceed as in Exercise 27. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}_1'(1) = \langle 1, -1, 2\rangle$ and $\mathbf{r}_2'(2) = \langle -1, 1, 4\rangle$. So $\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

Note: In Exercise 27, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

$$\frac{d}{dt}\left[\mathbf{u}(f(t))\right] = \frac{d}{dt}\left\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\right\rangle$$

$$= \left\langle \frac{d}{dt}\left[u_1(f(t))\right], \frac{d}{dt}\left[u_2(f(t))\right], \frac{d}{dt}\left[u_3(f(t))\right] \right\rangle$$

$$= \left\langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\right\rangle$$

$$= f'(t)\mathbf{u}'(t)$$

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41. $D_t\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$$= \left(-4t\,\mathbf{j} + 9t^2\,\mathbf{k}\right)\cdot\left(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}\right) + \left(\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k}\right)\cdot\left(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}\right)$$

$$= -4t\cos t + 9t^2\sin t + 1 + 2t^2\sin t + 3t^3\cos t$$

$$= 1 - 4t\cos t + 11t^2\sin t + 3t^3\cos t$$

42. $D_t\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$$= \left(-4t\,\mathbf{j} + 9t^2\,\mathbf{k}\right)\times\left(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}\right) + \left(\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k}\right)\times\left(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}\right)$$

$$= \left(-4t\sin t - 9t^2\cos t\right)\mathbf{i} + \left(9t^3 - 0\right)\mathbf{j} + \left(0 + 4t^2\right)\mathbf{k}$$

$$+ \left(-2t^2\cos t + 3t^3\sin t\right)\mathbf{i} + \left(3t^3 - \cos t\right)\mathbf{j} + \left(-\sin t + 2t^2\right)\mathbf{k}$$

$$= \left[(\sin t)\left(3t^3 - 4t\right) - 11t^2\cos t\right]\mathbf{i} + \left(12t^3 - \cos t\right)\mathbf{j} + \left(6t^2 - \sin t\right)\mathbf{k}$$

45. $\frac{d}{dt}\left|\mathbf{r}(t)\right| = \frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{1/2} = \frac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\left[2\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\left|\mathbf{r}(t)\right|}$