10.3 Arc Length and Curvature

1. $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\Big]_{-10}^{10} = 20\sqrt{29}$.

2. $\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \;\Rightarrow$ $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,|t| = \sqrt{5}\,t$ for $0 \le t \le \pi$. Then using Formula 3, we have $L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}\,t\,dt = \sqrt{5}\,\frac{t^2}{2}\Big]_0^\pi = \frac{\sqrt{5}}{2}\pi^2$.

3. $\mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \;\Rightarrow$ $|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$). Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

49. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi)\rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$\begin{aligned} L &= \int_0^{2.9\times 10^8 \times 2\pi} \left|\mathbf{r}'(t)\right| dt \\ &= \int_0^{2.9\times 10^8 \times 2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\,dt \\ &= \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\,t\,\Big]_0^{2.9\times 10^8 \times 2\pi} \\ &= 2.9 \times 10^8 \times 2\pi\sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2} \\ &\approx 2.07 \times 10^{10}\ \text{Å} \text{ — more than two meters!} \end{aligned}$$