10.5. PARAMETRIC SURFACES

$y = u_0 \cos u_0 \cos v$, $z = u_0 \sin v$.

11. $\mathbf{r}(u, v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = \cos v$, $y = \sin v$, $z = u$. Then $x^2 + y^2 = \cos^2 v + \sin^2 v = 1$ and $z = u$ with no restriction on u, so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z = u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z-axis.

12. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = u$. Then $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z^2$, which represents the equation of a cone with axis the z-axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2 + y^2 = u^2$ in the plane $z = u$. If v is constant, x, y, z are each scalar multiples of u, corresponding to the straight line grid curves through the origin.

13. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v, y = u\sin v, z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x = u^3$, $y = u\sin v$, $z = u\cos v$. Then $y^2 + z^2 = u^2\sin v^2 + u^2\cos v^2 = u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x = u^3$. The graph then must be graph III. If v is held constant, so $v = v_0$, we have $y = u\sin v_0$ and $z = u\cos v_0$. Then $y = (\tan v_0)\,z$, so the grid curves we see running lengthwise along the surface in the planes $y = kz$ correspond to keeping v constant.

15. $x = (u - \sin u)\cos v$, $y = (1 - \cos u)\sin v$, $z = u$. If u is held constant, x and y give an equation of an ellipse in the plane $z = u$, thus the grid curves are horizontally oriented ellipses. Note that when $u = 0$, the "ellipse" is the single point $(0,0,0)$, and when $u = \pi$, we have $y = 0$ while x ranges from $-\pi$ to π, a line segment parallel to the x-axis in the plane $z = \pi$. This is the upper "seam" we see in graph II. When v is held constant, $z = u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x = (1 - u)(3 + \cos v)\cos 4\pi u$, $y = (1 - u)(3 + \cos v)\sin 4\pi u$, $z = 3u + (1 - u)\sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2}\cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

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18. Solving the equation for z gives $z^2 = 1 - 2x^2 - 4y^2 \;\Rightarrow\; z = -\sqrt{1 - 2x^2 - 4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 - 2x^2 - 4y^2}$.

Alternate solution: The equation can be rewritten as $\dfrac{x^2}{(1/\sqrt{2})^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, and if we let $x = \dfrac{1}{\sqrt{2}}u\cos v$ and $y = \frac{1}{2}u\sin v$, then $z = -\sqrt{1 - 2x^2 - 4y^2} = -\sqrt{1 - u^2\cos^2 v - u^2\sin^2 v} = -\sqrt{1 - u^2}$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

21. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = 2$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $2 \le x^2 + y^2 \le 4$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.