11.4 Tangent Planes and Linear Approximations

4. $z = f(x,y) = y\ln x \quad \Rightarrow \quad f_x(x,y) = y/x,\ f_y(x,y) = \ln x$, so $f_x(1,4) = 4$, $f_y(1,4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1,4)(x-1) + f_y(1,4)(y-4) \quad \Rightarrow \quad z = 4(x-1) + 0(y-4)$ or $z = 4x - 4$.

12. $f(x,y) = \sin(2x+3y)$. The partial derivatives are $f_x(x,y) = 2\cos(2x+3y)$ and $f_y(x,y) = 3\cos(2x+3y)$, so $f_x(-3,2) = 2$ and $f_y(-3,2) = 3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is $L(x,y) = f(-3,2) + f_x(-3,2)(x+3) + f_y(-3,2)(y-2) = 0 + 2(x+3) + 3(y-2) = 2x + 3y$.

13. $f(x,y) = \sqrt{20 - x^2 - 7y^2} \quad \Rightarrow \quad f_x(x,y) = -\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y) = -\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1) = -\frac{2}{3}$ and $f_y(2,1) = -\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

$$\begin{aligned} f(x,y) &\approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 3 - \tfrac{2}{3}(x-2) - \tfrac{7}{3}(y-1) \\ &= -\tfrac{2}{3}x - \tfrac{7}{3}y + \tfrac{20}{3} \end{aligned}$$

Thus $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.84\overline{6}$.

16. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 11.3.4. Since $f_v(40,20) = \lim\limits_{h\to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table: $f_v(40,20) \approx \dfrac{f(50,20) - f(40,20)}{10} = \dfrac{40-28}{10} = 1.2$, $f_v(40,20) \approx \dfrac{f(30,20) - f(40,20)}{-10} = \dfrac{17-28}{-10} = 1.1$. Averaging these values gives $f_v(40,20) \approx 1.15$. Similarly, $f_t(40,20) = \lim\limits_{h\to 0} \dfrac{f(40,20) - f(40,20)}{h}$, so we use $h = 10$ and $h = -5$: $f_t(40,20) \approx \dfrac{f(40,30) - f(40,20)}{10} = \dfrac{31-28}{10} = 0.3$, $f_t(40,20) \approx \dfrac{f(40,15) - f(40,20)}{-5} = \dfrac{25-28}{-5} = 0.6$. Averaging these values gives $f_t(40,15) \approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v,t) &\approx f(40,20) + f_v(40,20)(v-40) + f_t(40,20)(t-20) \\ &\approx 28 + 1.15(v-40) + 0.45(t-20) \end{aligned}$$

When $v = 43$ and $t = 24$, we estimate $f(43,24) \approx 28 + 1.15(43-40) + 0.45(24-20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.