**8.** $f(x, y, z) = 8x - 4z$, $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z \rangle$.

Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But

$5 = x^2 + 10y^2 + z^2 = \left(\dfrac{4}{\lambda}\right)^2 + 10(0)^2 + \left(-\dfrac{2}{\lambda}\right)^2 \Rightarrow 5 = \dfrac{20}{\lambda^2} \Rightarrow \lambda = \pm 2$, so $f$ has possible extreme

values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of $f$ on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

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**10.** $f(x, y, z) = x^2 y^2 z^2$, $g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle$,

$\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $\nabla f = \lambda \nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of $f$ on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

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**12.** $f(x, y, z) = x^4 + y^4 + z^4$, $g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 4x^3, 4y^3, 4z^3 \rangle$,

$\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.

*Case 1:* If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$

yielding 8 points each with an $f$-value of $\frac{1}{3}$.

*Case 2:* If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are

equal with common value $\frac{1}{2}$ and the corresponding $f$-value is $\frac{1}{2}$.

*Case 3:* If exactly two of the variables are 0, then the third variable has value $\pm 1$ with corresponding $f$-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of $f$ is 1 and the minimum value is $\frac{1}{3}$.

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**23.** Let the sides of the rectangle be $x$ and $y$. Then $f(x, y) = xy$, $g(x, y) = 2x + 2y = p \Rightarrow \nabla f(x, y) = \langle y, x \rangle$,

$\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

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**29.** $f(x, y, z) = xyz$, $g(x, y, z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then

$\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

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**37.** We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints

$g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and

$\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need (1) $2x = \lambda + 2\mu x$, (2) $2y = \lambda + 2\mu y$, (3) $2z = 2\lambda - \mu$,

(4) $x + y + 2z = 2$, and (5) $x^2 + y^2 - z = 0$. From (1) and (2), $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$.

Putting this in (3) gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into (1) says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or

$z = -\frac{1}{2}$. Then (4) and (5) become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this

case gives no solution. So we must have $x = y$. Then (4) and (5) become $2x + 2z = 2$ and $2x^2 - z = 0$ which

imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$.

The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is

the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.