17. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2 - r^2}\, dr$$

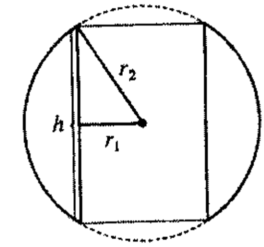
$$= 2[\theta]_0^{2\pi}\left[-\tfrac{1}{3}\left(a^2 - r^2\right)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

22. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V = 2\iint_{r_1^2\le x^2+y^2\le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r\, dr = \tfrac{4\pi}{3}\left[-\left(r_2^2 - r^2\right)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}\left(r_2^2 - r_1^2\right)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$. Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



24. By symmetry,

$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{1-\sin\theta} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\left[r^2\right]_{r=0}^{r=1-\sin\theta} d\theta = \int_{-\pi/2}^{\pi/2}\left(1 - 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[1 + \tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta = \int_{-\pi/2}^{\pi/2}\left(\tfrac{3}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta$$

since $2\sin\theta$ is an odd function. But $\frac{3}{2} - \frac{1}{2}\cos 2\theta$ is an even function, so $A = \int_0^{\pi/2}(3 - \cos 2\theta)\, d\theta = \left[3\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$.

29. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2 + y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \Rightarrow z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\, dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\, dr\, d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta = \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi}$$

$$= 1800\pi$$

Thus the pool contains $1800\pi \approx 5655\ \text{ft}^3$ of water.

30. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\, dr = [\theta]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$

$$= 2\pi\left[-Re^{-R} - e^{-R} + 0 + 1\right] = 2\pi\left(1 - Re^{-R} - e^{-R}\right)\ \text{ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 844.