12.6 Surface Area

2. $z = f(x,y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \le 9$, so by Formula 6

$$A(S) = \iint_D \sqrt{1 + (-2)^2 + (-5)^2}\, dA = \sqrt{30} \iint_D dA = \sqrt{30}\, A(D)$$
$$= \sqrt{30}\,(\pi \cdot 3^2) = 9\sqrt{30}\,\pi$$

3. $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\}$. Thus

$$A(S) = \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D$$
$$= \sqrt{14}\,(\tfrac{1}{2} \cdot 2 \cdot 3) = 3\sqrt{14}$$

5. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi}\int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr$$
$$= [\theta]_0^{2\pi}\left[\tfrac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

22. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x$, $y = \sqrt{1 - z^2}$, $z = z$ with $x^2 + z^2 \le 1$. Then $\mathbf{r}(x,z) = \langle x, \sqrt{1 - z^2}, z\rangle \Rightarrow \mathbf{r}_x = \langle 1, 0, 0\rangle$, $\mathbf{r}_z = \langle 0, -z/\sqrt{1 - z^2}, 1\rangle$ and $\mathbf{r}_x \times \mathbf{r}_z = \langle 0, -1, -z/\sqrt{1 - z^2}\rangle \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{1 + \frac{z^2}{1 - z^2}} = \frac{1}{\sqrt{1 - z^2}}$.

$$A(S) = \iint_{x^2 + z^2 \le 1} |\mathbf{r}_x \times \mathbf{r}_z|\, dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz$$
$$= 4\int_0^1\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz \quad \text{(by the symmetry of the surface)}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t \to 1^-} 4\int_0^t\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz = \lim_{t \to 1^-} 4\int_0^t \frac{\sqrt{1 - z^2}}{\sqrt{1 - z^2}}\, dz$$
$$= \lim_{t \to 1^-} 4\int_0^t dz = \lim_{t \to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4\,(4) = 16$.

Alternate Solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x^2 + z^2 \le 1 \Leftrightarrow x^2 + \sin^2\theta \le 1 \Leftrightarrow -\sqrt{1 - \sin^2\theta} \le x \le \sqrt{1 - \sin^2\theta} \Leftrightarrow -\cos\theta \le x \le \cos\theta$. Then $\mathbf{r}_x = \langle 1, 0, 0\rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta\rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta\rangle \Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = 1$, so $A(S) = \int_{-\pi/2}^{\pi/2}\int_{-\cos\theta}^{\cos\theta} 1\, dx\, d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta\, d\theta = 2\sin\theta\big]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.