

11. (a) \mathbf{F} has continuous first-order partial derivatives and $\frac{\partial}{\partial y} 2xy = 2x = \frac{\partial}{\partial x} (x^2)$ on \mathbb{R}^2 , which is open and simply-connected. Thus, \mathbf{F} is conservative by Theorem 6. Then we know that the line integral of \mathbf{F} is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C . Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f , so that $\nabla f = \mathbf{F}$. We know $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Integrating $f_x(x, y)$ with respect to x , we have $f(x, y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x, y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant. Thus $f(x, y) = x^2y + K$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2) - f(1, 2) = 18 - 2 = 16$ for each curve.

13. (a) $f_x(x, y) = x^3y^4$ implies $f(x, y) = \frac{1}{4}x^4y^4 + g(y)$ and $f_y(x, y) = x^4y^3 + g'(y)$. But $f_y(x, y) = x^4y^3$ so $g'(y) = 0 \Rightarrow g(y) = K$, a constant. We can take $K = 0$, so $f(x, y) = \frac{1}{4}x^4y^4$.

(b) The initial point of C is $\mathbf{r}(0) = (0, 1)$ and the terminal point is $\mathbf{r}(1) = (1, 2)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = 4 - 0 = 4$.

16. (a) $f_x(x, y, z) = 2xz + y^2$ implies $f(x, y, z) = x^2z + xy^2 + g(y, z)$ and so $f_y(x, y, z) = 2xy + g_y(y, z)$. But $f_y(x, y, z) = 2xy$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = x^2z + xy^2 + h(z)$ and $f_z(x, y, z) = x^2 + h'(z)$. But $f_z(x, y, z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \Rightarrow h(z) = z^3 + K$. Hence $f(x, y, z) = x^2z + xy^2 + z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0, 1, -1)$ and $t = 1$ corresponds to $(1, 2, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, -1) = 6 - (-1) = 7$.

We know that if the vector field (call it \mathbf{F}) is conservative, then around any closed path C , $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C , so the integral around C will be negative. Therefore the field is not conservative.

$$\nabla f(x, y) = \cos(x - 2y)\mathbf{i} - 2\cos(x - 2y)\mathbf{j}$$

- (a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0, 0) = \sin 0 = 0$ and $f(\pi, \pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0, 0)$ to (π, π) ; that is, $\mathbf{r}(t) = \pi t\mathbf{i} + \pi t\mathbf{j}$, $0 \leq t \leq 1$.

- (b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0, 0) = \sin 0 = 0$ and $f(\frac{\pi}{2}, 0) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2}t\mathbf{i}$, $0 \leq t \leq 1$, the straight line from $(0, 0)$ to $(\frac{\pi}{2}, 0)$.