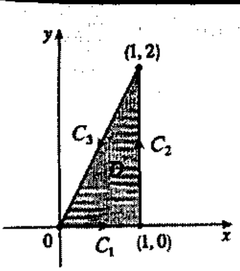
3. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\ y = 0 \;\Rightarrow\; dy = 0\,dt,\ 0 \le t \le 1.$

C_2: $x = 1 \;\Rightarrow\; dx = 0\,dt,\ y = t \;\Rightarrow\; dy = dt,\ 0 \le t \le 2.$

C_3: $x = 1 - t \;\Rightarrow\; dx = -dt,\ y = 2 - 2t \;\Rightarrow\; dy = -2\,dt,\ 0 \le t \le 1.$

Thus $\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$

$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$

$= 0 + \left[\frac{1}{4}t^4\right]_0^2 + \left[\frac{2}{3}(1-t)^3 + \frac{8}{3}(1-t)^6\right]_0^1 = 4 - \frac{10}{3} = \frac{2}{3}$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$= \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

9. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$

$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)\,dx\,dy = \int_0^1 \left(y^{1/2} - y^2\right) dy = \frac{1}{3}$

15. The region D enclosed by C is given, in polar coordinates, by $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 2\right\}$. Thus

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y^2 - x^2y)\,dx + xy^2\,dy = \iint_D (y^2 - 2y + x^2)\,dA$

$= \int_0^{\pi/4} \int_0^2 (r^2 - 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/4} \left[4 - \frac{16}{3}\sin\theta\right] d\theta$

$= \left[4\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/4} = \pi + \frac{8}{3}(\sqrt{2} - 2)$

20. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos t)\left(3\sin^2 t\cos t\right) dt = 3\int_0^{2\pi} \frac{1}{8}(1 - \cos 4t)\,dt = \frac{3}{4}\pi$

21. (a) Using Equation 13.2.8, we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$

$= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt$

$= \int_0^1 (x_1y_2 - x_2y_1)\,dt = x_1y_2 - x_2y_1$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

area of polygon $= A(D) = \iint_D dA = \frac{1}{2}\int_C x\,dy - y\,dx$

$= \frac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$

To evaluate these integrals we use the formula from (a) to get

$A(D) = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$.

(c) $A = \frac{1}{2}[(0\cdot1 - 2\cdot0) + (2\cdot3 - 1\cdot1) + (1\cdot2 - 0\cdot3) + (0\cdot1 - (-1)\cdot2) + (-1\cdot0 - 0\cdot1)]$

$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$