13.8 The Divergence Theorem

1. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

2. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

(b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \;\Rightarrow\; \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1+2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1+2y$ is negative, and P_2 is a sink.

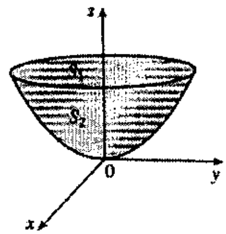
4. $\operatorname{div}\mathbf{F} = 8z$, so

$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 8zr\,dz\,dr\,d\theta = 2\pi\int_0^1 (4r - 4r^5)\,dr = \frac{8}{3}\pi$.

On S_1: $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3\pi$.

S_2: $\mathbf{F} = (x^3 + xy^2)\,\mathbf{i} + (y^3 + yx^2)\,\mathbf{j} + 3(x^2+y^2)^2\,\mathbf{k}$,

$-(\mathbf{r}_x \times \mathbf{r}_y) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$ and



$$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 1}\left(-x^4 - y^4 - 2x^2y^2\right)dA = -\int_0^{2\pi}\int_0^1 r^5\,dr\,d\theta = -\frac{\pi}{3}$$

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = 3\pi - \frac{\pi}{3} = \frac{8}{3}\pi$.

7. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2) = e^x\sin y - e^x\sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^2 2yz\,dz\,dy\,dx = 2\int_0^1 dx\int_0^1 y\,dy\int_0^1 z\,dz$$
$$= 2[x]_0^1\left[\tfrac{1}{2}y^2\right]_0^1\left[\tfrac{1}{2}z^2\right]_0^2 = 2$$

17. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D(-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\, r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since

$\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(z^2x) + \dfrac{\partial}{\partial y}\left(\tfrac{1}{3}y^3 + \tan z\right) + \dfrac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get

$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally

$\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

22. $\frac{1}{3}\iint_S\mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\iiint_E \operatorname{div}\mathbf{F}\,dV = \frac{1}{3}\iiint_E 3\,dV = V(E)$

23. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ by Theorem 13.5.11.