Problem Set 1 Solutions

1. As discussed in class, if each outcome of an experiment is equally likely then probability of an event is simply the number of outcomes "in" the event divided by the total number of outcomes. In other words, $\mathbb{P}(E) = N(E)/N(\Omega)$ where $N(S)$ denotes the number of elements in set S

Thus, each of the problems requires counting the number of elements in some set.

a) $N(A) = (3 \text{ face cards per suit})(4 \text{ suits})$

$= 12$

$\mathbb{P}(A) = 12/52$

b) $N(A \cap B) = 2$. The only face cards in B are the two red jacks. In other words, $A \cap B = \{x : x \text{ is a jack and } x \text{ is red}\}$.

$\mathbb{P}(A \cap B) = 2/52$.

c) $A \cup B = \{x : x \text{ is a face card } \textbf{or} \ x \text{ is a red } 9, 10, \text{jack}\}$.

There are 12 face cards as ~~we~~ shown in part a). ~~The~~ Since this includes the red jacks, we only need to ~~count~~ add the # of ~~the~~ red 9s, 10s in $A \cup B$ to get $N(A \cup B)$. Thus,

$N(A \cup B) = 12 + 4 = 16$, and $\mathbb{P}(A \cup B) = 16/52$.

1 d). $C \cup D = \Omega$ so $\mathbb{P}(C \cup D) = 1$.

e). $C \cap D = \emptyset$ so $\mathbb{P}(C \cap D) = 0$.

2. a)

HHHH	HHTH	THHH	THTH
HHHT	HHTT	THHT	THTT
HTHH	HTTH	TTHH	TTTH
HTHT	HTTT	TTHT	TTTT

In general, you should aim for a <u>systematic</u> way of identifying all the outcomes. I've assembled them in a 4×4 grid above which has a certain symmetry. You might also count them in groups that have the same number of tails. This would give you all $1+4+6+4+1 = 16$ outcomes.

b). As in problem 1, the assumption of equally likely outcomes reduces each of these computations to simply counting the number of elements in a set.

i). $N(A) = 5$. There are 4 outcomes with 3 heads and 1 outcome with all 4 heads. Thus, $\mathbb{P}(A) = \frac{N(A)}{N(\Omega)} = \frac{5}{16}$.

ii) Below

iii). $N(B) = 1+4+6 = 11$. There are 6 outcomes (or sequences) with ~~with~~ exactly 2 heads, another 4 with exactly 1 heads, and 1 outcome with 0 heads. Thus, $\mathbb{P}(B) = \frac{N(B)}{N(\Omega)} = \frac{11}{16}$

iii) $A \cap B = \emptyset$ so $\mathbb{P}(A \cap B) = 0$.

iv). $A \cup C = \{$ sequences with heads on the third toss or at least three heads $\}$.

There are 5 outcomes with at least three heads as we saw in part i). We need to add to this, the # of outcomes in which heads appears on the third toss, keeping in mind that we ~~might~~ have already accounted for some of these outcomes in our first five. Thus, we look for sequences with less than three heads, & with heads on the third toss. There's 1 such sequence with just one head and 3 more with 2 heads, (3 because there's 3 positions for the second heads). In all, we have $N(A \cup C) = 5+1+3 = 9$ so $\mathbb{P}(A \cup C) = 9/16$.

You might have noticed that all these words can be summed up in the following expressions:

$A \cap (A^c \cap C) = \emptyset$ and $A \cup (A^c \cap C) = A \cup C$ so

$$\mathbb{P}(A \cup C) = \mathbb{P}(A) + \mathbb{P}(A^c \cap C) \quad \text{by properties of } \mathbb{P}$$

$$= 5/16 + 4/16$$

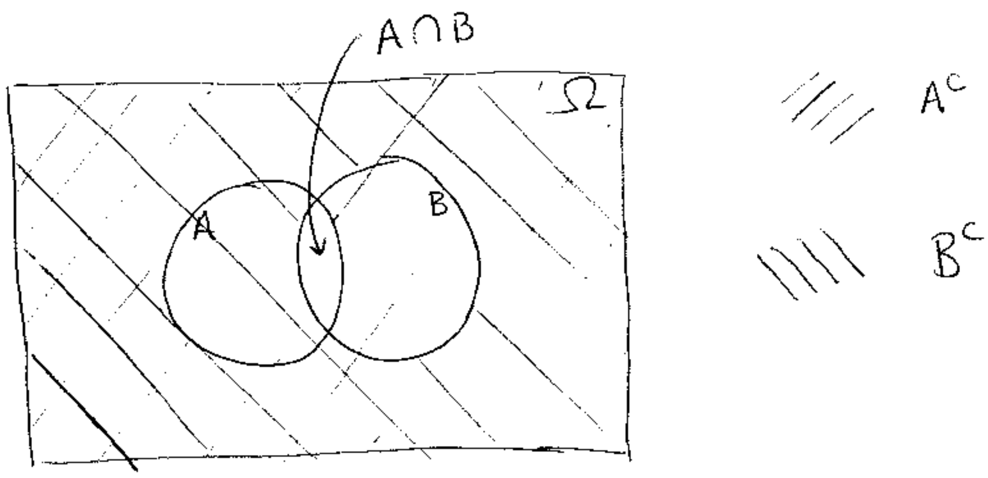
$$= 9/16.$$

v). $N(D) = 4$. There are 4 positions for the head that are possible. Thus, $\mathbb{P}(D) = 4/16$.

vi) Same as iv) due to typo.

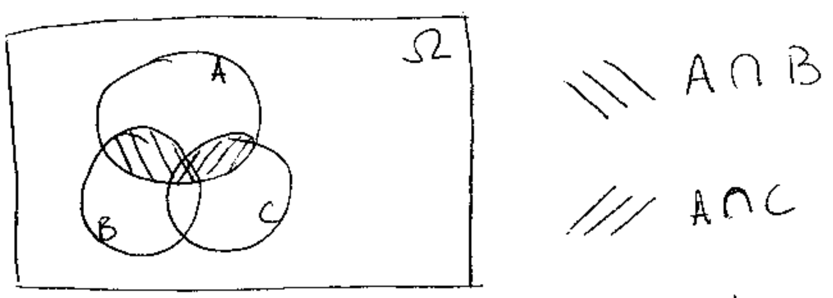
vii) Since $D \subset B$, $B \cup D = B$. Thus, $\mathbb{P}(B \cup D) = \frac{11}{16}$ as determined in part iii).

3. a)



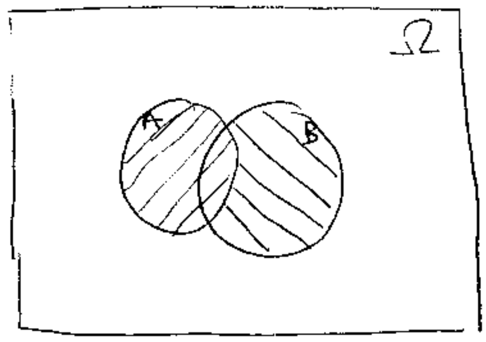
$(A\cap B)^c$ is everything not in $A \cap B$. But everything not in $A \cap B$ is hatched either /// or \\\ (or possibly both ways). In other words, $(A \cap B)^c = A^c \cup B^c$.

b.



Notice that $(A\cap B) \cup (A \cap C)$ includes the entire hatched region. This includes all points in A that are either in B or C.

4.



/// A

\\\ $B \cap A^c$

Notice that $A \cup B = A \cup (B \cap A^c)$ and that $A \cap (B \cap A^c) = \emptyset$. In other words, the two hatched regions together make up all of $A \cup B$ and they don't overlap. By the properties of the probability set function $\mathbb{P}$,

$$(*) \quad \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c).$$

Now, we do something similar for B. We use $B = (B \cap A) \cup (B \cap A^c)$ (see problem 3b if you don't understand this) and $(B \cap A) \cap (B \cap A^c) = \emptyset$ to conclude $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$. Solving for $\mathbb{P}(A^c \cap B)$ gives $\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Substituting in equation $(*)$ gives the desired result.

· Check this result for problem 1 just plugging in the numbers you calculated there. You will also need $\mathbb{P}(B) = 6/52$, $\mathbb{P}(C) = 13/52$, and $\mathbb{P}(D) = 39/52$. Notice that $\mathbb{P}(C \cup D) = \mathbb{P}(C) + \mathbb{P}(D)$ because C and D have empty intersection. In this case, this statement reduces to one of the properties of the probability set function.

5. We use our knowledge of the binomial distribution to calculate the probability of 350 hits with each rifle. Let's call rifle A the one which has a hit rate of $2/3$ and rifle B the other. The two probabilities are

$$p_A = \binom{600}{350}\left(\frac{2}{3}\right)^{350}\left(\frac{1}{3}\right)^{250} \quad \text{and}$$

$$p_B = \binom{600}{350}\left(\frac{1}{2}\right)^{350}\left(\frac{1}{2}\right)^{250}.$$

To decide which is bigger we compare their ratio to 1.

$$r = p_A / p_B$$

$$= \frac{\cancel{\binom{600}{350}}\left(\frac{2}{3}\right)^{350}\left(\frac{1}{3}\right)^{250}}{\cancel{\binom{600}{350}}\left(\frac{1}{2}\right)^{350}\left(\frac{1}{2}\right)^{250}} = \frac{2^{950}}{3^{600}}$$

Since these numbers in the numerator and denominator will not fit on a calculator, we compare $\log r$ to 0. If $\log r$ is positive then $p_A > p_B$ and if $\log r$ is negative then $p_A < p_B$.

$$\log r = \log 2^{950} - \log 3^{600}$$

$$= 950 \log 2 - 600 \log 3.$$

$$\approx 658.49 - 659.17$$

$$< 0.$$

Thus, $p_A < p_B$ and the maximum likelihood guess is that rifle B was used.