Problem Set 3 Solutions

1. a) $\mathbb{P}(T_+) = \mathbb{P}(T_+ \mid D)\,\mathbb{P}(D) + \mathbb{P}(T_+ \mid D^c)\,\mathbb{P}(D^c)$.

Here we've used the rule of averaged conditional probability with the partition D, D^c. In other words, since everyone either has the disease or does not have the disease (but not both), we can sum the probs. of testing positive given that someone does or does not have the disease. Doing this requires numbers for the right side above. We're given

$\mathbb{P}(T_+ \mid D^c) = .05$

$\mathbb{P}(T_- \mid D) = .2$

We use the fact that $\mathbb{P}(T_+ \mid D) + \mathbb{P}(T_- \mid D) = 1$ to conclude that

$\mathbb{P}(T_+ \mid D) = .8$.

Thus,

$\mathbb{P}(T_+) = (.8)(.01) + (.05)(.99) = .0575$.

b) $\mathbb{P}(D \cap T_-) = \mathbb{P}(T_- | D)\, \mathbb{P}(D)$ (defn. of cond. probability)

$= (.2)(.01) = .002$

c) $\mathbb{P}(D^c \cap T_-) = \mathbb{P}(T_- | D^c)\, \mathbb{P}(D^c)$.

$= (1 - \mathbb{P}(T_+ | D^c))\, \mathbb{P}(D^c)$

$= (.95)(.99) = .9405$

d) $$\mathbb{P}(D | T_+) = \frac{\mathbb{P}(D \cap T_+)}{\mathbb{P}(T_+)}$$

$$= \frac{\mathbb{P}(T_+ | D)\, \mathbb{P}(D)}{\left(\mathbb{P}(T_+ | D)\, \mathbb{P}(D) + \mathbb{P}(T_+ | D^c)\, \mathbb{P}(D^c)\right)}$$ (using 1a)

(This result is just the statement of Bayes' rule. Notice how easy it was to derive Bayes rule!)

$$\mathbb{P}(D | T_+) = \frac{(.8)(.01)}{(.8)(.01) + (.05)(.99)}$$

$\approx .139$ (or about a 13.9% chance).

e) This question is simply asking about the meaning of the probabilities calculated in this problem. For example, the 13.9 % ~~seus~~ probability calculated in part d) means that if we <u>repeated</u> this experiment many times, about 13.9% of the time a person who tested positive would actually have the disease.

2. a) $\mathbb{P}(L) = \mathbb{P}(L \mid G_{<20})\,\mathbb{P}(G_{<20}) + \mathbb{P}(L \mid G_{20-27})\,\mathbb{P}(G_{20-27})$
$+ \mathbb{P}(L \mid G_{28-36})\,\mathbb{P}(G_{28-36}) + \mathbb{P}(L \mid G_{>36})\,\mathbb{P}(G_{>36})$

Here we've again used the rule of averaged conditional probability with partition $G_{<20}, G_{20-27}, G_{28-36}, G_{>36}$.

$\mathbb{P}(L) = (.54)(.0004) + (.813)(.0059) + (.379)(.0855)$
$+ (.0354)(.9028)$

$\approx$ ~~xxxxx~~ $.069$

b) $\mathbb{P}(L \cap G_{\leq 27}) = \mathbb{P}(L \cap G_{<20}) + \mathbb{P}(L \cap G_{20-27})$

$$= \mathbb{P}(L \mid G_{<20})\,\mathbb{P}(G_{<20}) + \mathbb{P}(L \mid G_{20-27})\,\mathbb{P}(G_{20-27})$$

$$= (.54)(.0004) + (.813)(.0059).$$

$$\approx .00501$$

$$\mathbb{P}(L)\,\mathbb{P}(G_{\leq 27}) = \mathbb{P}(L)\left(\mathbb{P}(G_{<20}) + \mathbb{P}(G_{20-27})\right)$$

$$\approx \overset{(.069)}{\cancel{(.03796)}}\,(.0004 + .0059)$$

$$\approx \cancel{.000239}\quad .000435$$

Since $\mathbb{P}(L \cap G_{\leq 27}) \neq \mathbb{P}(L)\,\mathbb{P}(G_{\leq 27})$, these two events are not independent.

c) $\mathbb{P}(G_{\leq 36} \mid L) = 1 - \mathbb{P}(G_{>36} \mid L).$

$$\mathbb{P}(G_{>36} \mid L) = \frac{\mathbb{P}(G_{>36} \cap L)}{\mathbb{P}(L)}$$

$$= \frac{\mathbb{P}(L \mid G_{>36})\,\mathbb{P}(G_{>36})}{\mathbb{P}(L)}.$$

$$\approx \frac{(.035)(.9028)}{(.069)\cancel{(.03796)}}$$

$$\approx .458 \cancel{\ldots} \Rightarrow P(G \leq 36 \mid L) = .542$$

3. The outcome space for this experiment is $\{BB, WW, BW, WB\}$ where B, W denotes black and white balls respectively.

a) $\mathbb{P}(\{2^{nd} \text{ ball white}\}) = \mathbb{P}(\{WW, BW\})$

$= \mathbb{P}(\{WW\}) + \mathbb{P}(\{BW\})$.

$= \mathbb{P}(W_2 | W_1)\mathbb{P}(W_1) + \mathbb{P}(W_2 | B_1)\mathbb{P}(B_1)$.

where I'm now using W_i as shorthand for ~~to represent~~ drawing white on the i^{th} draw, eg $W_2 = \{2^{nd} \text{ ball white}\} = \{WW, BW\}$.

$$= \left(\frac{7}{13}\right)\left(\frac{4}{10}\right) + \left(\frac{4}{13}\right)\left(\frac{6}{10}\right)$$

$$= \frac{28}{130} + \frac{24}{130} = \frac{52}{130} = \frac{2}{5}$$

b) $\mathbb{P}(B_1 | W_2) = \dfrac{\mathbb{P}(B_1 \cap W_2)}{\mathbb{P}(W_2)}$

$= \dfrac{\mathbb{P}(W_2 | B_1)\mathbb{P}(B_1)}{\mathbb{P}(W_2)}$ (Bayes' rule again!)

$$= \frac{24/130}{52/130} = \frac{24}{52} = \frac{6}{13}$$

If you tried doing 3b) without something like Bayes' rule, you probably found out the hard way how handy Bayes is.

c). $\mathbb{P}(W_2) = \mathbb{P}(W_2 | W_1)\, \mathbb{P}(W_1) + \mathbb{P}(W_2 | B_1)\, \mathbb{P}(B_1)$

as in part a).

$$\mathbb{P}(W_2) = \left(\frac{w+d}{w+b+d}\right)\left(\frac{w}{w+b}\right) + \left(\frac{w}{w+b+d}\right)\left(\frac{b}{w+b}\right)$$

$$= \frac{w}{w+b}\left(\frac{w+d}{w+d+b} + \frac{b}{w+d+b}\right)$$

$$= \frac{w}{w+b}$$

4 a) First we need to think about what it means to find the probability distribution function (or prob. density function) for Z. We want to find the possible values Z can take and what the prob. are for each of those values. Let S_Z be the space of random variable Z. Then

$S_Z = \{0, 1, 4, 9, \ldots, N^2\}$.

This follows directly from the fact that X takes on the values $0, \dots, N$. Now

$$\mathbb{P}(Z=0) = \mathbb{P}(X=0) = \frac{1}{N+1}$$

$$\mathbb{P}(Z=1) = \mathbb{P}(X=1) = \frac{1}{N+1}$$

$$\mathbb{P}(Z=4) = \mathbb{P}(X=2) = \frac{1}{N+1}$$

$$\mathbb{P}(Z=k^2) = \mathbb{P}(X=k) = \frac{1}{N+1}.$$

Thus, I can define $f_Z(z) = \mathbb{P}(Z=z)$. I've just shown that

$$f_Z(z) = \begin{cases} \frac{1}{N+1} & \text{if } z=k^2 \text{ for } k=0,\dots,N \\ 0 & \text{otherwise.} \end{cases}$$

b) As in part a), we begin by finding the space of possible values of W. Since X can be $0, \dots, N$ and Y can be $0, 1$ only, we know that W can be $0, \dots, N+1$ only. That is, $S_W = \{0, \dots, N+1\}$.

$$\mathbb{P}(W=0) = \mathbb{P}(X=0 \ \& \ Y=0)$$

$$= \mathbb{P}(X=0)\,\mathbb{P}(Y=0)$$ because X, Y are independent.

$$= \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right).$$

$$\mathbb{P}(W=1) = \mathbb{P}(X=1 \ \& \ Y=0) + \mathbb{P}(X=0 \ \& \ Y=1)$$

$$= \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{N+1}.$$

$$\mathbb{P}(W=k) = \mathbb{P}(X=k \ \& \ Y=0) + \mathbb{P}(X=k-1 \ \& \ Y=1)$$

for $k = 1, \ldots, N$

$$= \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right) = \frac{1}{N+1}.$$

For the value $N+1$ of W, there's again only one way to get it, just like 0.

$$\mathbb{P}(W=N+1) = \mathbb{P}(X=N \ \& \ Y=1)$$

$$= \left(\frac{1}{N+1}\right)\left(\frac{1}{2}\right)$$

Thus,
$$f_W(\omega) = \begin{cases} \frac{1}{2}\left(\frac{1}{N+1}\right) & \text{for } \omega=0, \omega=N+1 \\ \frac{1}{N+1} & \text{for } \omega=1,\ldots,N \\ 0 & \text{otherwise.} \end{cases}$$

is the pdf for W.