

Problem Set 4 Solutions

4c (from pset 3). Using the same notation,

$$E[Z] = \sum z f_Z(z)$$

$$= \sum_{k=0}^N k^2 f_Z(k^2)$$

because $f_Z(z) = 0$ for all other z .

$$= \sum_{k=0}^N k^2 \left(\frac{1}{N+1} \right)$$

$$= \frac{1}{N+1} \left(\frac{N(N+1)(2N+1)}{6} \right)$$

$$= \frac{N(2N+1)}{6}$$

$$E[W] = \sum w f_W(w)$$

$$= \left(\sum_{w=1}^N w \left(\frac{1}{N+1} \right) \right) + 0 \frac{1}{2} \left(\frac{1}{N+1} \right) + (N+1) \frac{1}{2} \left(\frac{1}{N+1} \right)$$

$$= \frac{1}{N+1} \left(\frac{N(N+1)}{2} \right) + \frac{1}{2}$$

$$= \frac{N}{2} + \frac{1}{2}$$

$$= \frac{N+1}{2}$$

1. a. At most \$1. The expected winnings in such a game are \$1 also if you pay less than that to play, you would make money per game (on average). We now calculate this result for expected winnings. Let X be the random variable representing the number of heads that appear in one game.

$$E[X] = \sum_{k=0}^3 k f_X(k).$$

$$= \sum_{k=0}^3 k \cdot \binom{3}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{3-k} \quad \text{because we know } X \text{ has a binomial distrib.}$$

$$= 1 \left(\frac{3 \cdot 4}{27}\right) + 2 \left(\frac{3 \cdot 2}{27}\right) + 3 \left(\frac{1 \cdot 1}{27}\right)$$

$$= \frac{27}{27} = 1$$

Thus, we expect 1 heads per game and so \$1 of winnings.

$$b. \quad E[X] = \sum_{k=0}^n k \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

$$= \frac{1}{3^n} \sum_{k=0}^n k 2^{n-k} \binom{n}{k}$$

To evaluate this sum, we have to use some ingenuity. We go back to the statement of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Now we take the partial of both sides with respect to x :

$$n(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^{k-1} y^{n-k}$$

Notice that the right side of this equation has the general form we're interested in.

We just need to evaluate it at $x=1, y=2$:

$$n(1+2)^{n-1} = \sum_{k=0}^n \binom{n}{k} k 2^{n-k}$$

Thus, the sum we're evaluating simplifies:

$$E[X] = \frac{1}{3^n} (n \cdot 3^{n-1}) = \frac{n}{3}$$

Note that this is what we might have expected naively. If we flip n times and the probability of heads on each flip is $\frac{1}{3}$, then in the long run we'd expect $\frac{n}{3}$ heads.

2. The PDFs for X and Y are.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Since X and Y are independent, their joint PDF is the product of their PDFs:

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

Now $P(X^2 + Y^2 \leq R^2)$ is the probability that (X, Y) lies inside a circle of radius R . In other words,

$$P(X^2 + Y^2 \leq R^2) = \iint_{D_R} f(x, y) dA.$$

Given the shape of the region and the form of the function, it makes sense to use polar coordinates.

$$P(X^2 + Y^2 \leq R^2) = \int_0^{2\pi} \int_0^R \frac{1}{2\pi} e^{-\frac{r^2}{2}} r dr d\theta.$$

$$= \int_0^{2\pi} \left[\int_0^{R^2} e^{-u} du \right] \frac{1}{2\pi} d\theta.$$

$$= \int_0^{2\pi} \left(-e^{-u} \Big|_0^{R^2} \right) \frac{1}{2\pi} d\theta$$

$$= \int_0^{2\pi} \left(1 - e^{-R^2} \right) \frac{1}{2\pi} d\theta.$$

$$= \left(1 - e^{-R^2} \right)$$

19. (a) $f(x, y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 1] \times [0, 2]$, we can say

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_0^1 \int_0^2 Cx(1+y) dy dx \\ &= C \int_0^1 x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=2} dx = C \int_0^1 4x dx = C [2x^2]_0^1 = 2C\end{aligned}$$

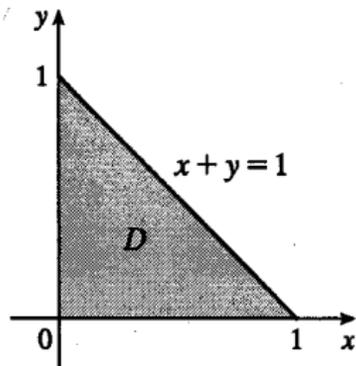
Then $2C = 1 \Rightarrow C = \frac{1}{2}$.

(b) $P(X \leq 1, Y \leq 1) = \int_{-\infty}^1 \int_{-\infty}^1 f(x, y) dy dx = \int_0^1 \int_0^1 \frac{1}{2}x(1+y) dy dx$

$$= \int_0^1 \frac{1}{2}x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{2}x \left(\frac{3}{2} \right) dx = \frac{3}{4} \left[\frac{1}{2}x^2 \right]_0^1 = \frac{3}{8} \text{ or } 0.375$$

- (c) $P(X + Y \leq 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus

$$\begin{aligned}P(X + Y \leq 1) &= \iint_D f(x, y) dA = \int_0^1 \int_0^{1-x} \frac{1}{2}x(1+y) dy dx \\ &= \int_0^1 \frac{1}{2}x \left[y + \frac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{2}x \left(\frac{1}{2}x^2 - 2x + \frac{3}{2} \right) dx \\ &= \frac{1}{4} \int_0^1 (x^3 - 4x^2 + 3x) dx = \frac{1}{4} \left[\frac{x^4}{4} - 4\frac{x^3}{3} + 3\frac{x^2}{2} \right]_0^1 \\ &= \frac{5}{48} \approx 0.1042\end{aligned}$$



22. (a) Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x, y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

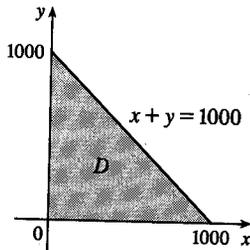
$$\begin{aligned} P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x, y) dy dx \\ &= \int_0^{1000} \int_0^{1000} 10^{-6}e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000} dx \int_0^{1000} e^{-y/1000} dy \\ &= 10^{-6} \left[-1000e^{-x/1000} \right]_0^{1000} \left[-1000e^{-y/1000} \right]_0^{1000} \\ &= (e^{-1} - 1)^2 \approx 0.3996 \end{aligned}$$

- (b) Now we are asked for the probability that the combined

lifetimes of both bulbs is 1000 hours or less. Thus we want

to find $P(X + Y \leq 1000)$, or equivalently $P((X, Y) \in D)$

where D is the triangular region shown in the figure. Then



$$\begin{aligned} P(X + Y \leq 1000) &= \iint_D f(x, y) dA = \int_0^{1000} \int_0^{1000-x} 10^{-6}e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} \left[-1000e^{-(x+y)/1000} \right]_{y=0}^{y=1000-x} dx = -10^{-3} \int_0^{1000} (e^{-1} - e^{-x/1000}) dx \\ &= -10^{-3} \left[e^{-1}x + 1000e^{-x/1000} \right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$

25. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P , and $k dA$ is the number of infected individuals in an element of area dA , then $f(P, A)k dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA . Integration over D gives the number of infections of the person at A due to all the infected people in D . In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

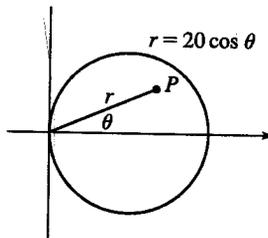
$$\begin{aligned} E &= \iint_D kf(P, A) dA = k \iint_D \frac{20 - d(P, A)}{20} dA \\ &= k \iint_D \left[1 - \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{20} \right] dx dy \end{aligned}$$

- (b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k \iint_D \left[1 - \frac{1}{20} \sqrt{x^2 + y^2} \right] dx dy = k \int_0^{2\pi} \int_0^{10} \left(1 - \frac{r}{20} \right) r dr d\theta \\ &= 2\pi k \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_0^{10} = 2\pi k \left(50 - \frac{50}{3} \right) = \frac{200}{3} \pi k \approx 209k \end{aligned}$$

For A at the edge of the city, it is convenient to use a polar coordinate system centered at A . Then the polar equation for the circular boundary of the city becomes $r = 20 \cos \theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k \int_{-\pi/2}^{\pi/2} \int_0^{20 \cos \theta} \left(1 - \frac{r}{20} \right) r dr d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_{r=0}^{r=20 \cos \theta} d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \left(200 \cos^2 \theta - \frac{400}{3} \cos^3 \theta \right) d\theta \\ &= 200k \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2} \cos 2\theta - \frac{2}{3} (1 - \sin^2 \theta) \cos \theta \right] d\theta \\ &= 200k \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta - \frac{2}{3} \sin \theta + \frac{2}{3} \cdot \frac{1}{3} \sin^3 \theta \right]_{-\pi/2}^{\pi/2} \\ &= 200k \left[\frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} + \frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} \right] \\ &= 200k \left(\frac{\pi}{2} - \frac{8}{9} \right) \approx 136k \end{aligned}$$



Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.