

§9.3 The Dot Product

18. $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ are orthogonal when $\langle -6, b, 2 \rangle \cdot \langle b, b^2, b \rangle = 0 \Leftrightarrow (-6)(b) + (b)(b^2) + (2)(b) = 0$
 $\Leftrightarrow b^3 - 4b = 0 \Leftrightarrow b(b+2)(b-2) = 0 \Leftrightarrow b = 0$ or $b = \pm 2$.

25. $(\text{orth}_a \mathbf{b}) \cdot \mathbf{a} = (\mathbf{b} - \text{proj}_a \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - (\text{proj}_a \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \mathbf{a} \cdot \mathbf{a}$

$$\mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} |\mathbf{a}|^2 = \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} = 0$$

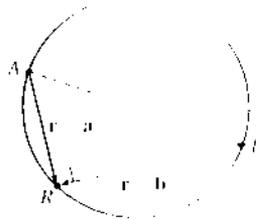
So they are orthogonal by (2).

33. Let us note that $\mathbf{n} = \langle a, b \rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ are on the line, then $\mathbf{n} \cdot (Q_1 - Q_2) = aa_1 - aa_2 + bb_1 - bb_2 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line. Let $P_1 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection

$$\text{of } P_1 \text{ onto } \mathbf{n}, \text{comp}_{\mathbf{n}}(\overrightarrow{P_1 P_2}) = \frac{|\mathbf{n} \cdot \langle x_2 - x_1, y_2 - y_1 \rangle|}{|\mathbf{n}|} = \frac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

since $ax_2 + by_2 = -c$. The required distance is $\frac{|3 \cdot -2 + -4 \cdot 3 + 5|}{\sqrt{3^2 + 4^2}} = \frac{13}{5}$.

34. $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ implies that the vectors $\mathbf{r} - \mathbf{a}$ and $\mathbf{r} - \mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B . The center of this circle is the midpoint of AB , that is,



$$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \left\langle \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2), \frac{1}{2}(a_3 + b_3) \right\rangle, \text{ and its radius is}$$

$$\frac{1}{2} |\mathbf{a} - \mathbf{b}| = \frac{1}{2} \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

Or: Expand the given equation, substitute $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

37. Consider the H-C-H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1, 0, 0)$ and $(0, 1, 0)$ (or any H-C-H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\langle 1 - \frac{1}{2}, 0 - \frac{1}{2}, 0 - \frac{1}{2} \rangle = \langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle$ and $\langle 0 - \frac{1}{2}, 1 - \frac{1}{2}, 0 - \frac{1}{2} \rangle = \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle$. The bond angle, θ , is therefore given by

$$\cos \theta = \frac{\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle}{|\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle| |\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle|} = \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}} \sqrt{\frac{3}{4}}} = -\frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^\circ.$$

§9.4 The Cross Product

8. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 3 \end{vmatrix} \mathbf{k}$

$$= (2 - 6)\mathbf{i} - (-3 - 12)\mathbf{j} + (-9 - 12)\mathbf{k} = -4\mathbf{i} + 15\mathbf{j} - 21\mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle -4, 15, -21 \rangle \cdot \langle -3, 2, 2 \rangle = 12 + 30 - 42 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} .

and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle -4, 15, -21 \rangle \cdot \langle 6, 3, 1 \rangle = -24 + 45 - 21 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{b} .

Assignment 3

p. 2

$$9. \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} \mathbf{k}$$

$$= (3t^4 - 2t^4) \mathbf{i} - (3t^3 - t^3) \mathbf{j} + (2t^2 - t^2) \mathbf{k} = t^4 \mathbf{i} - 2t^3 \mathbf{j} + t^2 \mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle t, t^2, t^3 \rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} .

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{b} .

16. The parallelogram is determined by the vectors $\overrightarrow{KL} = \langle 0, 1, 3 \rangle$

and $\overrightarrow{KN} = \langle 2, 5, 0 \rangle$, so the area of parallelogram $KLMN$ is

$$|\overrightarrow{KL} \times \overrightarrow{KN}| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} \right| = |(-15)\mathbf{i} - (-6)\mathbf{j} + (-2)\mathbf{k}| = |-15\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}| = \sqrt{265} \approx 16.28.$$

18. (a) $\overrightarrow{PQ} = \langle 1, 1, 3 \rangle$ and $\overrightarrow{PR} = \langle 3, 2, 5 \rangle$, so a vector orthogonal to the plane through P , Q , and R is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(5) - (3)(2), (3)(3) - (1)(5), (1)(2) - (1)(3) \rangle = \langle -1, 4, -1 \rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by \overrightarrow{PQ} and \overrightarrow{PR} is

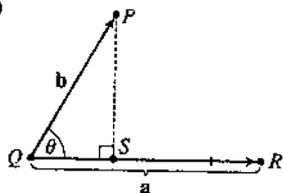
$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |(-1, 4, -1)| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}$, so the area of triangle PQR is

$$\frac{1}{2} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{2}.$$

$$22. \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -6 - 9 - 4 = -19.$$

So the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} and \mathbf{c} is $|-19| = 19$ cubic units.

27. (i)



The distance between a point and a line is the length of the perpendicular

from the point to the line, here $|\overrightarrow{PS}| = d$. But referring to triangle

PQS , $d = |\overrightarrow{PS}| = |\overrightarrow{QP}| \sin \theta = |\mathbf{b}| \sin \theta$. But θ is the angle between

$\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by definition of the cross product,

$$\sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} \text{ and so } d = |\mathbf{b}| \sin \theta = \frac{|\mathbf{b}| |\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}.$$

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, -2, -1 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1, -5, -7 \rangle$. Then

$\mathbf{a} \times \mathbf{b} = \langle (-2)(-7) - (-1)(-5), (-1)(1) - (-1)(-7), (-1)(-5) - (-2)(1) \rangle = \langle 9, -8, 7 \rangle$. Thus the

distance is $d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}} \sqrt{81 + 64 + 49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$.