SOLUTIONS : STOKES' THEOREM

SECTION ONE

(1) $\quad \text{curl } \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & 1 & z \end{vmatrix} = \langle 0, 0, -3y^2 \rangle$

Parametrize S by $\quad x = u \quad y = v \quad z = u^2 + v^2$
for $(u,v) \in R = \{(u,v) : u^2+v^2 \leq 1\}$.
Then
$$\underset{\sim}{r}_u = \langle 1, 0, 2u \rangle$$
$$\underset{\sim}{r}_v = \langle 0, 1, 2v \rangle$$
$$\underset{\sim}{r}_u \times \underset{\sim}{r}_v = \langle -2u, -2v, 1 \rangle$$

and so
$$\iint_S (\text{curl } \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \iint_R \langle 0, 0, -3v^2 \rangle \cdot \langle -2u, -2v, 1 \rangle \, du\,dv$$
$$= -\iint_R 3v^2 \, du\,dv = -\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} 3r^3 \sin^2\theta \, d\theta\, dr$$
$$= -3\pi/4$$

(2) $\quad \text{curl } \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \langle 1, 1, 1 \rangle$

Parametrize S by thinking about spherical polar co-ordinates:

$$x = 2\sin u \cos v \qquad y = 2 \sin u \sin v \qquad z = 2\cos u$$

for $\quad 0 \leq u \leq \pi/2, \quad 0 \leq v \leq 2\pi$

Then
$$\underset{\sim}{r}_u = \langle 2\cos u \cos v, \ 2\cos u \sin v, \ -2\sin u \rangle$$
$$\underset{\sim}{r}_v = \langle -2\sin u \sin v, \ 2\sin u \cos v, \ 0 \rangle$$
$$\underset{\sim}{r}_u \times \underset{\sim}{r}_v = \langle 4\sin^2 u \cos v, \ 4 \sin^2 u \sin v, \ 2\sin 2u \rangle$$

and so
$$\iint_S (\text{curl } \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \int_{u=0}^{u=\pi/2} \int_{v=0}^{v=2\pi} 4\sin^2 u (\underbrace{\cancel{\sin v} + \cos v}_{\text{symmetry}}) + 2\sin 2u \, dv\, du$$
$$= 4\pi \left[\tfrac{1}{2} \cos 2u \right]_{u=\pi/2}^{u=0} = 4\pi$$