SOLUTIONS: SURFACE INTEGRALS

SECTION ONE

(1) Parametrize S by $x = u \quad y = v \quad z = v + 3$
for $0 \le u^2 + v^2 \le 1$.
Then $\underline{r}_u = \langle 1, 0, 0 \rangle$, $\underline{r}_v = \langle 0, 1, 1 \rangle$ and $\underline{r}_u \times \underline{r}_v = \langle 0, -1, 1 \rangle$
so

$$\iint_S yz \, dS = \iint_R (v^2 + 3v) . \sqrt{2} \, du dv$$

where R is the disc $0 \le u^2 + v^2 \le 1$. Using polar co-ordinates in the uv-plane, we see that

$$\iint_S yz \, dS = \sqrt{2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^3 \sin^2\theta + 3r^2 \sin\theta \, dr \, d\theta$$

$$= \sqrt{2} . \frac{\pi}{4} = \underline{\underline{\frac{\pi}{2\sqrt{2}}}}$$

(2) Parametrize the funnel by $x = u$, $y = v$, $z = \sqrt{u^2+v^2}$
for $(u,v) \in R = \{ 1 \le \sqrt{u^2+v^2} \le 4 \}$.
Then $\underline{r}_u = \langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \rangle$, $\underline{r}_v = \langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \rangle$

$$\underline{r}_u \times \underline{r}_v = \left\langle -\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1 \right\rangle$$

$$\Rightarrow |\underline{r}_u \times \underline{r}_v| = \sqrt{2}$$

and so

$$\text{mass} = \iint_S (10 - z) \, dS = \iint_R (10 - \sqrt{u^2+v^2}) . \sqrt{2} \, dA$$

$$= \int_{r=1}^{r=4} \int_{\theta=0}^{\theta=2\pi} (10 - r) . \sqrt{2} \, r \, d\theta \, dr$$

$$= 2\sqrt{2}\pi \left[5r^2 - r^3/3 \right]_{r=1}^{r=4}$$

$$= \underline{\underline{108\sqrt{2}\pi}}$$