SECTION THREE

Suppose that our smooth simply-connected surface is not orientable. Try to choose an orientation, i.e. a smoothly-varying choice of outward-pointing normal vector. We fail. This means that there is a closed loop C on my surface so that the orientation reverses along this loop.

But the loop C bounds a disc (this is what simply-connected means) and this vector field only extends to a vector field over the disc which vanishes somewhere. But the normal vector to our surface doesn't vanish, as our surface is smooth. Thus we have a contradiction, and so S is orientable

An example of an orientable non-simply-connected surface is a torus (the surface of a bagel). This is orientable — it has a global outward-pointing normal vector field — and (since the loop around the "hole" in the bagel does not bound a disc) it is not simply-connected.