EVERYTHING IS AN OPTIMIZATION QUESTION: THE EULER–LAGRANGE EQUATIONS

1. Refresher : Newton's Laws

Let us think about a particle of mass m moving in 3-dimensional space, whose position at time t is

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Suppose that the particle moves under the influence of a conservative force with potential $V(x, y, z)$, so that when it is at the point (x, y, z) it experiences a force of

$$-\nabla V(x, y, z)$$

(1) Newton's Second Law gives an equation describing the motion of the particle. Write it down.

(2) Deduce that the motion of the particle is such that the *energy*

$$\frac{m\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t)}{2} + V\left(x(t), y(t), z(t)\right)$$

is conserved. Here $\dot{}$ indicates the derivative with respect to t.

2. The Lagrangian picture of classical mechanics

Newton's Laws give a complete description of classical mechanics, but this formulation has several drawbacks:

- it is a vector equation, so its form depends on the co-ordinate system that you use
- it requires that you know all the forces acting on the particle explicitly (rather than, for example, knowing that some of the forces constrain the motion of the particle to lie on the surface of a sphere but not knowing exactly what they are)
- it is *local* : it describes the behaviour of a physical system at each particular instant in time. Often it would be more useful to describe the behaviour for all times at once.
- it does not apply to physical systems which do not consist of particles (such as the electromagnetic field)
- it gives no insight into how to *quantize* classical mechanics[1]

The Lagrangian formulation of classical mechanics has none of these problems. The idea is to pick out the path $\mathbf{r}(t)$ of the particle as the minimum of some function on the space of all possible particle paths. A physical system is specified by a function $L(x, y, z, \dot{x}, \dot{y}, \dot{z})$ called the *Lagrangian*. For the particle which we considered above, this will be

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{m}{2} \langle \dot{x}, \dot{y}, \dot{z} \rangle \cdot \langle \dot{x}, \dot{y}, \dot{z} \rangle - V\left(x, y, z\right)$$

[1] Newton can be forgiven for not noticing this drawback — he was working in the 1680s whereas quantum mechanics was not invented until the 1920s.

Choose an initial time t_1, an initial point P, a final time t_2 and a final point Q. We are going to find the path followed by our particle which starts at P at time t_1 and ends at Q at time t_2. To do this we define a function on the set of paths, called the *action*, as follows. If $\gamma(t)$ is the

$$\gamma(t) = \langle x(t), y(t), z(t) \rangle$$

then define the action $S(\gamma)$ to be

$$S(\gamma) = \int_{t_1}^{t_2} L(x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t))\, dt$$

The actual path followed by our particle will be the *minimum* of this function[2].

3. The Euler-Lagrange equations

We want to minimize the action $S(\gamma)$. At a minimum[3] γ, if we perturb γ a tiny bit then there will be no first-order change in S. So, imagine perturbing the path γ by taking

$$x(t) \rightsquigarrow x(t) + \epsilon a(t)$$

where ϵ is very small and $a(t)$ is any function of t which vanishes at t_1 and t_2. The reason for this last condition is that we want our perturbed path still to start at P and end at Q. Show that the change in S, to first order in ϵ, is

$$\int_{t_1}^{t_2} \left(a(t)\frac{\partial L}{\partial x} + \dot{a}(t)\frac{\partial L}{\partial \dot{x}} \right)\, dt$$

This is zero. Deduce that

$$\int_{t_1}^{t_2} a(t) \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)\, dt = 0$$

Now show that if $h(t)$ is a function such that

$$\int_{t_1}^{t_2} h(t) f(t)\, dt = 0$$

for all continuous functions f then $h(t) = 0$ for all t with $t_1 \leq t \leq t_2$.

[2]If there is no minimum, then if the particle starts at P at time t_1 it will not end up at Q at time t_2!

[3]Or indeed at any critical point.

Deduce that

(1)
$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

We got this equation by perturbing only the x-component of the path γ. What equations do you get by perturbing the y- and z-components of γ?

Work out what these equations say when

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{m}{2} \langle \dot{x}, \dot{y}, \dot{z} \rangle \cdot \langle \dot{x}, \dot{y}, \dot{z} \rangle - V(x, y, z)$$

Consider the case of the *free particle*, when $V(x, y, z) = 0$. Deduce that the particle moves such that

$$\frac{\partial L}{\partial \dot{x}} \qquad \frac{\partial L}{\partial \dot{y}} \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{z}}$$

are all constant. (*hint: look at (1)!*) What does this mean physically?