SOLUTIONS: POLAR CO-ORDINATES

SECTION ONE

(1) $\text{Volume} = \iint_{\text{base}} (\text{height})\, dA$

Here the base of the region is the disc $x^2+y^2 \le 4$ and the height is $(10-x^2-y^2)-6 = 4-x^2-y^2$, so

$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4-r^2)\, r\, dr\, d\theta$$

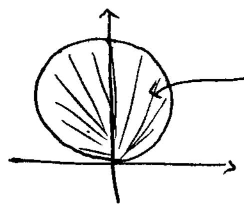
$$= 2\pi \left[2r^2 - \tfrac{1}{3}r^3\right]_{r=0}^{r=2} = \underline{\underline{32\pi/3}}$$

(2) $\text{Volume} = \iint_{\text{base}} (\text{height})\, dA$

The base of this region is the disc of radius 1 centred at $(0,1)$, which has equation $x^2 + (y-1)^2 \le 1 \Rightarrow x^2+y^2 \le 2y \Rightarrow r^2 \le 2r\sin\theta$

The height is $1+x+y$, so

$$\text{Volume} = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} (1 + r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$



$$= \int_{\theta=0}^{\theta=\pi} 2\sin^2\theta + \tfrac{8}{3}\sin^3\theta\cos\theta + \tfrac{8}{3}\sin^4\theta \; d\theta$$

$$= \pi + 0 + \pi = \underline{\underline{2\pi}}$$

SECTION TWO

(1) At $(4,0,2\pi)$, $u=4$ and $v=2\pi$. A normal vector to the tangent plane is

$$\underline{r}_u \times \underline{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle$$
$$= \langle \sin v, -\cos v, u\rangle$$

and at $u=4, v=2\pi$ this is $\langle 0,-1,4\rangle$. The tangent plane at $(4,0,2\pi)$ is therefore $4(z-2\pi) - (y-0) = 0$

i.e. $\underline{\underline{4z - y = 8\pi}}$

$$\text{Surface area} = \int_{u=0}^{u=1}\int_{v=0}^{v=2\pi} |\underline{r}_u \times \underline{r}_v| \, dv\, du$$

$$= \int_{u=0}^{u=1}\int_{v=0}^{v=2\pi} \sqrt{1+u}\; dv\, du$$

$$= \left[2\pi (1+u)^{3/2} \cdot \frac{2}{3}\right]_{u=0}^{u=1} = \underline{\underline{\frac{4\pi}{3}\left(2\sqrt{2}-1\right)}}$$

(2) Parametrize the surface by $\underline{r}(u,v) = \langle u, v, uv\rangle$ for (u,v) in the region $R = \{(u,v) : u^2+v^2 \leq 1\}$. Then $\underline{r}_u \times \underline{r}_v = \langle 1,0,v\rangle \times \langle 0,1,u\rangle$

$$= \langle -v, -u, 1\rangle$$

so surface area $= \iint_R \sqrt{1+u^2+v^2}\; dA$

$$= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} \sqrt{1+r^2}\; r\, d\theta\, dr = \left[\frac{2\pi}{3}(1+r^2)^{3/2}\right]_{r=0}^{r=1}$$

$$= \underline{\underline{\frac{2\pi}{3}\left(2\sqrt{2}-1\right)}}$$

(3) Since $x = \frac{1}{2}y + \frac{1}{2}z^2$, if $0 \leq y \leq 1$ and $0 \leq z \leq 1$ then $0 \leq x \leq 1$ automatically. Thus we may parametrize the surface as

$$\underline{r}(u,v) = \left\langle \tfrac{1}{2}u + \tfrac{1}{2}v^2, u, v\right\rangle \qquad 0 \leq u \leq 1,\ 0 \leq v \leq 1$$

We have $\underline{r}_u \times \underline{r}_v = \langle \frac{1}{2}, 1, 0\rangle \times \langle v, 0, 1\rangle = \langle 1, -\frac{1}{2}, -v\rangle$

so surface area $= \int_{u=0}^{u=1}\int_{v=0}^{v=1} \sqrt{1+1/4+v^2}\; dv\, du$

$= \ldots$ substitute $v = \frac{\sqrt{5}}{2}\sinh t \ldots$

$$= \underline{\underline{3/4 + \frac{5\ln(5)}{16}}}$$