SOLUTIONS : SURFACE AREA / TRIPLE INTEGRALS

SECTION ONE

(1) Parametrize the surface by $\underline{r}(u,v) = \langle u^2 - v^2, u, v \rangle$

where $(u,v) \in R = \{(u,v) : u^2 + v^2 \le 1\}$.

We have $\underline{r}_u \times \underline{r}_v = \langle 2u, 1, 0 \rangle \times \langle -2v, 0, 1 \rangle = \langle 1, -2u, 2v \rangle$

so surface area $= \iint_R \sqrt{1+4u^2+4v^2}\, dA$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \sqrt{1+4r^2}\; r\, dr\, d\theta = \left[2\pi \cdot \tfrac{1}{12} (1+4r^2)^{3/2} \right]_{r=0}^{r=1}$$

$$= \frac{\pi}{6} (5\sqrt{5} - 1)$$

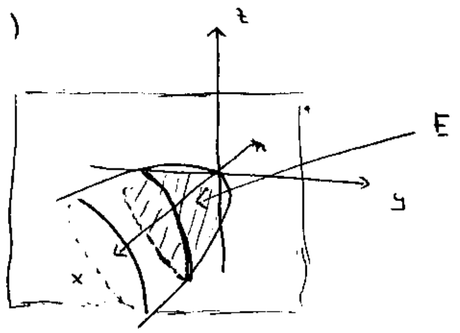
(2) Parametrize the surface by $\underline{r}(x,y) = \langle x, y, f(x,y) \rangle$, $(x,y) \in D$.

Then $\underline{r}_x \times \underline{r}_y = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle$

so surface area $= \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$

SECTION TWO

(1)



$x = 4y^2 + 4z^2$

$$\iiint_E x\, dV = \iint_D \left(\int_{x=4y^2+4z^2}^{x=4} x\, dx \right) dy\, dz$$

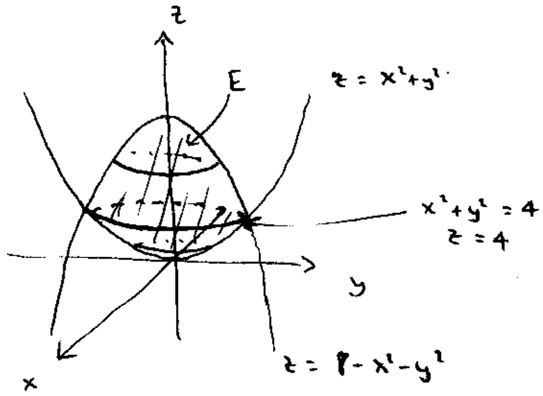
where D is the disc $y^2+z^2 \le 1$ in the yz-plane

$$= \iint_D \left[\tfrac{1}{2} x^2 \right]_{x=4y^2+4z^2}^{x=4} dy\, dz$$

$$= \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (8 - 8r^4)\, r\, d\theta\, dr$$

$$= 16\pi \left(\tfrac{1}{2} - \tfrac{1}{6} \right) = \frac{16\pi}{3}$$

(2)



$$\text{Mass} = \iiint_E x^2 \, dV$$

$$= \iint_D \left(\int_{z=x^2+y^2}^{z=8-x^2-y^2} x^2 \, dz \right) dx\,dy$$

where D is the disc $x^2+y^2 \leq 4$ in the xy-plane

$$\text{so} \quad \text{Mass} = \iint_D x^2(8-2x^2-2y^2) \, dx\,dy$$

$$= \int_{r=0}^{r=2} \int_{\theta=0}^{\theta=2\pi} r^2 \cos^2\theta \, (8-2r^2) \, r \, d\theta \, dr$$

$$= \int_{r=0}^{r=2} (8r^3-2r^5) \cdot \pi \, dr \quad = \pi \left[2r^4 - r^6/3 \right]_{r=0}^{r=2}$$

$$= \underline{\underline{\frac{32\pi}{3}}}$$

SECTION THREE

$$\iint_R f(x,y) \, dA = \iint_S f(g(u,v), h(u,v)) \left| \begin{matrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{matrix} \right| du\,dv$$

See Section 12.9 of the textbook.