SOLUTIONS : LINE INTEGRALS

SECTION ONE

(1) Parametrize the wire by $x = t^2$, $y = t$ for $0 \le t \le 2$.

$$\text{Mass of wire} = \int_C y \, ds$$

$$= \int_{t=0}^{t=2} t \sqrt{(2t)^2 + 1^2} \, dt$$

$$= \left[\frac{1}{12} \left(1 + 4t^2\right)^{3/2} \right]_{t=0}^{t=2} = \frac{1}{12}\left(17\sqrt{17} - 1\right)$$

(2) Parametrize the curve by $x = t$, $y = t^3$ for $0 \le t \le 1$.

$$\text{Area} = \int_C z \, ds = \int_{t=0}^{t=1} 36t^3 \sqrt{1^2 + (3t^2)^2} \, dt$$

$$= \int_{t=0}^{t=1} 36t^3 \sqrt{1 + 9t^4} \, dt$$

$$= \left[2/3 \left(1 + 9t^4\right)^{3/2} \right]_{t=0}^{t=1} = 2/3 \left(10\sqrt{10} - 1\right)$$

SECTION TWO

(1)
$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{r}'(t) \, dt = \int_{t=0}^{t=2} (t^7 \underset{\sim}{i} - t \underset{\sim}{j}) \cdot (3t^2 \underset{\sim}{i} + 2t \underset{\sim}{j}) \, dt$$

$$= \int_{t=0}^{t=2} 3t^9 - 2t^2 \, dt$$

$$= \left[\frac{3t^{10}}{10} - \frac{2}{3} t^3 \right]_{t=0}^{t=2} = \frac{3072}{10} - \frac{16}{3}$$

(2) Parametrize C by $\underset{\sim}{r}(t) = \langle t, 1-t^2 \rangle$, $-1 \leq t \leq 1$.

Then $\underset{\sim}{F}(\underset{\sim}{r}(t)) = \langle t^2, (1-t^2)^2 \rangle$

and $\underset{\sim}{r}'(t) = \langle 1, -2t \rangle$

so
$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{t=-1}^{t=1} \langle t^2, (1-t^2)^2 \rangle \cdot \langle 1, -2t \rangle \, dt$$
$$= \int_{t=-1}^{t=1} t^2 - 2t(1-t^2)^2 \, dt$$
$$= \underline{\underline{2/3}}$$

(3)
$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$
$$= \int_{t=a}^{t=b} \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) dt$$
$$= \int_{t=a}^{t=b} \frac{d}{dt} \big(f(x(t), y(t)) \big) \, dt \qquad \text{(Chain Rule)}$$
$$= \underline{\underline{f(x(b), y(b)) - f(x(a), y(a))}} \qquad \left(\begin{matrix}\text{Fundamental} \\ \text{Theorem of} \\ \text{Calculus}\end{matrix}\right)$$