SECTION ONE

(1) Put $f(x,y,z) = xe^{yz}$

The normal vector to the tangent plane is

$$\nabla f(1,0,5) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle \Big|_{(1,0,5)}$$

$$= \langle 1, 5, 0 \rangle$$

so the tangent plane is

$$(x-1) + 5(y-0) + 0(z-5) = 0$$

i.e. $\underline{\underline{x + 5y = 1}}$

(2) The direction vector of the normal line is

$$\nabla(x^2 - y^2 + 2z^2) = \langle 2x, -2y, 4z \rangle$$

and we need this to be parallel to the vector

$$\langle 5,3,6 \rangle - \langle 3,-1,0 \rangle = \langle 2,4,6 \rangle$$

Thus $\langle 2x, -2y, 4z \rangle = t\langle 2,4,6 \rangle$

for some t, so

$$x = t$$
$$y = -2t$$
$$z = 3/2\, t$$

Also $x^2 - y^2 + 2z^2 = 1$, so $t^2 - 4t^2 + 2 \cdot \frac{9t^2}{4} = 1$

$\Rightarrow \quad \frac{3}{2}t^2 = 1$

$\Rightarrow \quad t = \pm\sqrt{2/3}$

The points in question are

$\left(\sqrt{2/3},\ -2\sqrt{2/3},\ 4\sqrt{2/3}\right)$ and $\left(-\sqrt{2/3},\ 2\sqrt{2/3},\ -4\sqrt{2/3}\right)$

SECTION TWO

(1) Critical points satisfy $f_x = f_y = 0$, i.e.

$$\begin{cases} 3x^2y + 24x = 0 & \cdots (1) \\ x^3 - 8 = 0 & \cdots (2) \end{cases}$$

From (2), $x = 2$. Substituting into (1) we find

$12y + 48 = 0$

$\Rightarrow y = -4$

The only critical point is $(2, -4)$.

$f_{xx} = 6xy + 24 \qquad f_{xy} = f_{yx} = 3x^2 \qquad f_{yy} = 0$

so $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = -9x^4$

This is negative at $(2,-4)$, so the critical point is a saddle.

(2) Critical points satisfy $f_x = f_y = 0$, so

$$\begin{cases} (y - 2x^2y)\,e^{-x^2-y^2} = 0 & \cdots \text{①} \\ (x - 2xy^2)\,e^{-x^2-y^2} = 0 & \cdots \text{②} \end{cases}$$

Since $e^{-x^2-y^2}$ is never zero, we have

$$\begin{cases} y(1-2x^2) = 0 & \cdots \text{③} \\ x(1-2y^2) = 0 & \cdots \text{④} \end{cases}$$

From ③ either $y=0$ or $2x^2=1$

Case I: $y=0$

Then from ④ we have $x=0$ $\rightsquigarrow$ $\underline{\underline{(0,0)}}$

Case II: $2x^2=1$, so $x = \pm\sqrt{\frac{1}{2}}$

From ④ we have $2y^2=1$ also, so

$y = \frac{1}{\sqrt{2}}$ or $y = -1/\sqrt{2}$.

~~[illegible]~~

Critical points are : $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$

$$f_{xx} = 2xy(2x^2-3)\, e^{-x^2-y^2}$$

$$f_{xy} = f_{yx} = (2x^2-1)(2y^2-1)\, e^{-x^2-y^2}$$

$$f_{yy} = 2xy(2y^2-3)\, e^{-x^2-y^2}$$

so
$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= -e^{-2(x^2-y^2)} \left(1 - 4x^2 + 4x^4 - 4y^2 - 20x^2y^2 + 8x^4y^2 + 4y^4 + 8x^2y^4 \right)$$
$$= -e^{-2(x^2-y^2)} \left(1 - 4(x^2+y^2) + 4x^4 + 4y^4 + 8x^2y^2(x^2+y^2) - 20x^2y^2 \right)$$

$D(0,0) = -1$, $D\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 4$

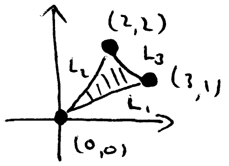
f_{xx} is positive at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
and negative at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

so we have a

saddle at $(0,0)$
local maxima at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
local minima at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -1/\sqrt{2}\right)$

SECTION THREE

(1)



There are no ~~[illegible]~~ critical points of f, so the max or min. must occur on the boundary. Parametrize the boundary:

L_1 is $\langle x,y\rangle = \langle 3t, t\rangle \quad 0\le t\le 1$

L_2 is $\langle x,y\rangle = \langle 2t, 2t\rangle \quad 0\le t\le 1$

L_3 is $\langle x,y\rangle = \langle 2+t, 2-t\rangle \quad 0\le t\le 1$

so f on L_1 is $f(3t,t) = 7t+5 \quad 0\le t\le 1$

maximized at $t=1$ i.e. $(3,1)$

f on L_2 is $f(2t,2t) = 2t+5 \quad 0\le t\le 1$

maximized at $t=1$ i.e. $(2,2)$

f on L_3 is $f(2+t,2-t) = 5t+7 \quad 0\le t\le 1$

maximized at $t=1$ i.e. $(3,1)$

The absolute maximum is at $(3,1)$, and the max. value is $\underline{\underline{f(3,1) = 12}}$

(2) Step 1: Find interior maxima/minima. These occur at critical points

$$\begin{cases} g_x = 0 \\ g_y = 0 \end{cases} \quad \text{gives} \quad \begin{cases} 2x+1 = 0 \\ 2y+1 = 0 \end{cases}$$

So only interior crit. point is $(-\frac{1}{2}, -\frac{1}{2})$.

$$g(-1/2, -1/2) = -1/2$$

Step 2: Find boundary extrema

The boundary $x^2+y^2=1$ is parametrized by

$$x(t) = \cos t \qquad y(t) = \sin t \qquad 0 \le t < 2\pi$$

and $f(x(t), y(t)) = 1 + \cos t + \sin t$

This is extremized when $\frac{d}{dt}(1 + \cos t + \sin t) = 0$

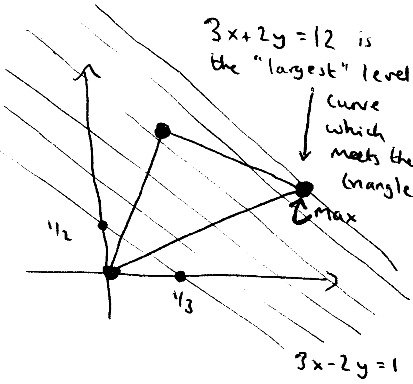
$$\Rightarrow \quad -\sin t + \cos t = 0$$

$$\Rightarrow \quad \tan t = 1 \qquad \text{so} \quad t = \pi/4, \ 3\pi/4$$

$$f(x(\pi/4), y(\pi/4)) = 1 + \sqrt{2} \qquad f(x(\tfrac{3\pi}{4}), y(\tfrac{3\pi}{4})) = 1 - \sqrt{2}$$

So the maximum value of $f(x,y)$ on the disc $x^2+y^2 \le 1$ is

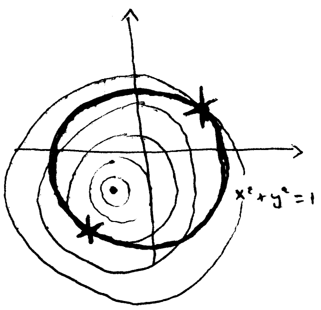
$$\max(-1/2, 1+\sqrt{2}, 1-\sqrt{2}) = \underline{\underline{1+\sqrt{2}}}.$$



SECTION FOUR

(1) (a) Level curves of f are $3x - 2y = \text{const.}$

(b)



$g(x,y) = (x+\frac{1}{2})^2 + (y+\frac{1}{2})^2 - \frac{1}{2}$

so level curves of g are circles centered at $(-1/2, -1/2)$

level curves of g are tangent to $x^2+y^2=1$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. The maximum clearly occurs at $(1/\sqrt{2}, 1/\sqrt{2})$ and equals $g(1/\sqrt{2}, 1/\sqrt{2}) = 1+\sqrt{2}$.

(c) Solve $g(a,b) = h(a,b)$ and $\nabla g(a,b) = t\,\nabla h(a,b)$ for some t ...