SOLUTIONS: MAXIMIZING AND MINIMIZING II

SECTION ONE

1 (a) Solve $f_x = f_y = 0$, i.e. $\begin{cases} 2x + y^2 = 0 & \cdots \text{①} \\ 2y + 2xy = 0 & \cdots \text{②} \end{cases}$

From ② either $y = 0$ or $x = -1$

Case I: $y = 0$

Then from ①, $2x = 0$ so the critical point is $(0,0)$

Case II: $x = -1$

Then from ①, $y^2 = 2$ so we have critical points $(-1, \sqrt{2})$
$(-1, -\sqrt{2})$

$f_{xx} = 2$ $f_{xy} = 2y$ $f_{yy} = 2x$

so $D = \begin{vmatrix} 2 & 2y \\ 2y & 2x \end{vmatrix} = 4x - 4y^2$

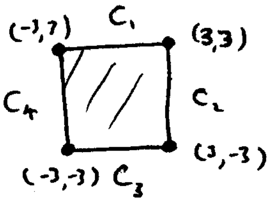
D is zero at $(0,0)$, so we don't know from this test what the critical point is, and is negative at $(-1, \pm\sqrt{2})$ so these are saddle points.

Plotting level curves on Mathematica (use Contour Plot[ ]) suggests that the critical point at $(0,0)$ is a local minimum; to prove this notice that

$$f(x,y) = x^2 + y^2(1+x)$$

is always positive for (x,y) near ~~[illegible]~~ $(0,0)$.

(b)



The curves making up the boundary can be parametrized as

C_1 : $x = t$, $y = 3$ $\quad -3 \le t \le 3$

C_2 : $x = 3$, $y = t$ $\quad -3 \le t \le 3$

C_3 : $x = t$, $y = -3$ $\quad -3 \le t \le 3$

C_4 : $x = -3$, $y = t$ $\quad -3 \le t \le 3$

On C_1 , $f(x(t), y(t)) = t^2 + 9t + 9$

$$= (t + 9/2)^2 + - \frac{45}{4}$$

is maximized at $t = 3$ (i.e. $(3,3)$) and minimized at $t = -3$ (i.e. $(-3,3)$)

Similarly, on C_2 $\quad f(x(t), y(t)) = 9 + 4t^2$

is maximized at $t = \pm 3$ (i.e. at $(+3,-3)$ and $(+3,3)$) and minimized at $t = 0$ i.e. $(3,0)$.

On C_3 a similar calculation shows that $f(x,y)$ is minimized at $(-3,-3)$ and maximized at $(3,3)$

On C_4 , $f(x,y)$ is maximized at $(-3,0)$ and minimized at $(-3,3)$ and $(-3,-3)$.

<u>Candidate maxima</u> : $(3,3) \rightsquigarrow f = 45$ $\quad (+3,-3) \rightsquigarrow f = 45$

$(-3,0) \rightsquigarrow f = 9$

<u>Candidate minima</u> : $(-3,3) \rightsquigarrow f = -9$ $\quad (3,0) \rightsquigarrow f = 9$

$(-3,-3) \rightsquigarrow f = -9$

so boundary maxima occur at $(3,3)$ and $(3,-3)$ (max value $= 45$)

and boundary minima occur at $(-3,3)$ and $(-3,-3)$ (Min. value $= -9$)

(c) Since there are no interior maxima, the global maxima on the square are ~~the~~ the boundary maxima: the maximum value that f attains on the square is 45.

The only interior minimum is at $(0,0)$ and $f(0,0)=0$ which is larger than the boundary minima, so the minimum value that f attains on the square is -9.

SECTION TWO:

(1) Put $f(x,y) = x^2+y^2$ $g(x,y) = x^4+y^4$

and solve $\nabla f = \lambda \nabla g$ and $g=1$:

$$\begin{cases} 2x = 4\lambda x^3 & \dots (1) \\ 2y = 4\lambda y^3 & \dots (2) \\ x^4+y^4 = 1 & \dots (3) \end{cases}$$

From (1) either $x=0$ or $4\lambda x^2 = 2$

Case I: $x=0$

From (3), $y = \pm 1$ so $(0,1)$ or $(0,-1)$

Case II: $4\lambda x^2 = 2$

From (2) either $y=0$ or $4\lambda y^2 = 2$

Case IIa: $y=0$

Then from (3), $x = \pm 1$ so $(1,0)$ or $(-1,0)$

Case IIb : $4\lambda y^2 = 2$

Since also $4\lambda x^2 = 2$ (we're in case II)

we have $4\lambda x^2 = 4\lambda y^2$

$\Rightarrow \quad x^2 = y^2$ (since $\lambda \neq 0$, because $4\lambda y^2 = 2$)

So $x = y$ or $x = -y$

Case IIb1 : $x = y$

Substituting into ③ we find $x = y = \sqrt[4]{2}$ $(\sqrt[4]{2}, \sqrt[4]{2})$

or $x = y = -\sqrt[4]{2}$ $(-\sqrt[4]{2}, -\sqrt[4]{2})$

Case IIb2 : $x = -y$

Substituting into ③ we find $x = \pm\sqrt[4]{2}$, so

$(\sqrt[4]{2}, -\sqrt[4]{2})$ or $(-\sqrt[4]{2}, \sqrt[4]{2})$

Possible extreme values : $f(0,1) = f(1,0) = f(0,-1) = f(-1,0) = 1$

$f(\pm\sqrt[4]{2}, \pm\sqrt[4]{2}) = 2\sqrt{2}$

so Constrained ~~maxima~~ minima at $(\pm 1, 0)$ and $(0, \pm 1)$, min. value $= 1$

constrained maxima at $(\pm\sqrt[4]{2}, \pm\sqrt[4]{2})$, max. value $= 2\sqrt{2}$.

(2) We want to minimize $x^2+y^2+z^2$

subject to $x^2y^2z = 1$

Put $f(x,y,z) = x^2+y^2+z^2$, $g(x,y,z) = x^2y^2z$ and solve $\nabla f = \lambda \nabla g$ and $g = 1$.

This is
$$\begin{cases} 2x = \lambda\, 2xy^2z & \cdots \text{①} \\ 2y = \lambda\, 2x^2yz & \cdots \text{②} \\ 2z = \lambda x^2y^2 & \cdots \text{③} \\ x^2y^2z = 1 & \cdots \text{④} \end{cases}$$

From ④ none of x, y, z are zero, so $\lambda \neq 0$ (from ①) and

$$\frac{1}{\lambda} = y^2z \quad \cdots \text{⑤}$$
$$\frac{1}{\lambda} = x^2z \quad \cdots \text{⑥}$$

Since $z \neq 0$, ⑤ and ⑥ give $x^2 = y^2$ ... ⑦

From ③, $\frac{1}{\lambda} = \frac{x^2y^2}{z}$ and comparing with ⑥ we find $\frac{x^2y^2}{z} = x^2z$ and hence $z^2 = y^2$ ... ⑧

From ⑧, $z = \pm y$.

<u>Case I</u>: $z = y$

Then from ④, and ⑦, $y^5 = 1$ so $y = 1$.

Thus $z = 1$ and (from (7)) $x = \pm 1$, so $(\pm 1, 1, 1)$.

Case II : $z = -y$

Then from (4) and (7), $-y^5 = 1$

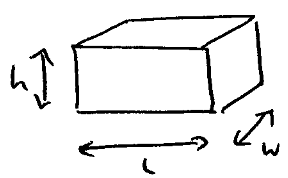
$\Rightarrow \quad y = -1$

$\Rightarrow \quad z = 1$

Now from (7), $x = \pm 1$, so $(\pm 1, -1, 1)$.

All these points are the same distance from the origin; the closest points are $(\pm 1, 1, 1)$ and $(\pm 1, -1, 1)$.

3) A similar argument, maximizing lwh



subject to $2lw + 2lh + 2wh = 64$

yields $l = w = h = \dfrac{8}{\sqrt{6}}$

and a volume of $\dfrac{256}{3\sqrt{6}}$