SOLUTIONS : LAGRANGE MULTIPLIERS

SECTION ONE

(1) We want to maximize xyz
subject to $2x + 3y + 5z = 7$

Put $f(x,y,z) = xyz$, $g(x,y,z) = 2x + 3y + 5z$ and solve:

$$\nabla f = \lambda \nabla g$$

and $g(x,y,z) = 7$

or in other words

$$yz = 2\lambda \quad \ldots \quad (1)$$
$$xz = 3\lambda \quad \ldots \quad (2)$$
$$xy = 5\lambda \quad \ldots \quad (3)$$
$$2x + 3y + 5z = 7 \quad \ldots \quad (4)$$

From (1) and (2) we see that none of x, y and z are zero.

From (1) and (2) we find $\frac{yz}{2} = \frac{xz}{3}$, and so $\underbrace{3y = 2x}_{(5)}$

From (1) and (3) we find $\frac{yz}{2} = \frac{xy}{5}$, and so $\underbrace{5z = 2x}_{(6)}$

Substituting (5) and (6) into (4) gives $6x = 7$, so

$$x = 7/6 \qquad y = 7/9 \qquad z = 7/15$$

and the maximum volume is $xyz = \underline{\underline{343/810}}$

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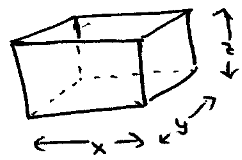
(2) 

We want to $\cancel{\text{max}}$minimize $f(x,y,z) = xy + 2xz + 2yz$

subject to $g(x,y,z) = xyz = 1$

Solve $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 1$, i.e.

$$y + 2z = \lambda yz \quad \cdots \text{ (1)}$$
$$x + 2z = \lambda xz \quad \cdots \text{ (2)}$$
$$2x + 2y = \lambda xy \quad \cdots \text{ (3)}$$
$$xyz = 1 \quad \cdots \text{ (4)}$$

From (4), none of x, y and z are zero.

From (1) and (2), $$\frac{y+2z}{yz} = \frac{x+2z}{xz}$$

$$\Rightarrow \quad xy + 2xz = xy + 2yz$$
$$\Rightarrow \quad x = y \quad \cdots \text{ (5)}$$

From (1) and (3) $$\frac{y+2z}{yz} = \frac{2x+2y}{xy}$$

$$\Rightarrow \quad xy + 2xz = 2xz + 2yz$$
$$\Rightarrow \quad x = 2z \quad \cdots \text{ (6)}$$

Substituting (5) and (6) into (4) we find $\dfrac{x^3}{2} = 1$

$\Rightarrow \quad x = \sqrt[3]{2} \qquad y = \sqrt[3]{2} \qquad z = \dfrac{\sqrt[3]{2}}{2}$

and the minimum surface area is $xy + 2xz + 2yz = \underline{\underline{3 \cdot 2^{2/3}}}$

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(3) (a) See the discussion at the end of section 11.8 of the textbook

(b) We solve

$$\nabla f = \lambda \nabla g + \mu \nabla h$$
$$g(x,y,z) = 1$$
$$h(x,y,z) = 4$$

where $g(x,y,z) = x+y+z$ and $h(x,y,z) = y^2+z^2$.

In other words:

$$1 = \lambda + 2\mu \cdot 0 \quad \cdots \quad (1)$$
$$2 = \lambda + 2\mu y \quad \cdots \quad (2)$$
$$0 = \lambda + 2\mu z \quad \cdots \quad (3)$$
$$x+y+z = 1 \quad \cdots \quad (4)$$
$$y^2+z^2 = 4 \quad \cdots \quad (5)$$

From (1), $\lambda = 1$. Substituting this into (2) and (3) we find

$$2\mu y = 1$$
$$2\mu z = -1$$

Thus $y = -z \quad \cdots \quad (6)$

Substituting (6) into (5) gives $y^2 = 2$, so either $y = \sqrt{2}$ or $y = -\sqrt{2}$.

Case I: $y = \sqrt{2}$, $z = -\sqrt{2}$. From (4), $x = 1$.

$$f(x,y,z) = 1+2\sqrt{2}$$

Case II: $y = -\sqrt{2}$, $z = \sqrt{2}$. From (4), $x = 1$.

$$f(x,y,z) = 1-2\sqrt{2}$$

Thus the constrained maximum occurs at $(1, \sqrt{2}, -\sqrt{2})$ and

SECTION TWO

(1) The tangent plane to the sphere at the point (x,y,z) has normal vector $\nabla(x^2+y^2+z^2) = \langle 2x, 2y, 2z\rangle$.

We need this to be parallel to $\langle 4,1,0\rangle$

so we need $\langle 2x, 2y, 2z\rangle = \lambda\langle 4,1,0\rangle$

$\Rightarrow \quad x = 2\lambda \quad y = \lambda/2 \quad$ and $z=0$

Also, $x^2+y^2+z^2=9$ so $4\lambda^2 + \lambda^2/4 = 9$

$$\Rightarrow \lambda^2 = \frac{9.4}{17}$$

$$\Rightarrow \lambda = \pm\frac{6}{\sqrt{17}}$$

The points where the tangent plane to $x^2+y^2+z^2=9$ is parallel to $4x+y=5$ are $\left(\frac{12}{\sqrt{17}}, \frac{3}{\sqrt{17}}, 0\right)$

and $\left(-\frac{12}{\sqrt{17}}, -\frac{3}{\sqrt{17}}, 0\right)$

(2) Max rate of change $= \|\nabla f\| = \left\|\left\langle 3x^2y + \frac{1}{2}\sqrt{\frac{y}{x}},\ x^3 + \frac{1}{2}\sqrt{\frac{x}{y}}\right\rangle\right\|$

At $(2,1)$ this is $\left\|\left\langle 12 + \frac{1}{2\sqrt{2}},\ 8+\sqrt{2}\right\rangle\right\| = \sqrt{\frac{9409}{64} + (8+\sqrt{2})^2}$

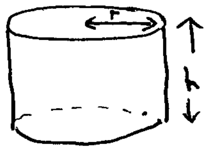
$$\approx 15.35$$

It occurs in the direction of ∇f, i.e. in the direction of

$$\frac{1}{\sqrt{\frac{9409}{64} + (8+\sqrt{2})^2}}\left\langle 12 + \frac{1}{2\sqrt{2}},\ 8+\sqrt{2}\right\rangle.$$

Rate of change of f in the direction of $\langle 1,1\rangle$ $= \nabla f \cdot \langle 1,1\rangle = \underline{\underline{20 + \frac{5}{2\sqrt{2}}}}$

(3)



Let volume = V.

$V = \pi r^2 h$

so $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= 2\pi r h.\,1 + \pi r^2(-2) = \underline{\underline{0}}$$ when $r = h = 2$

(4) First find critical points by solving $f_x = f_y = 0$.

$$f_x = 2e^{-x^2-y^2}(3-3x^2+y^2)\,x$$
$$f_y = 2e^{-x^2-y^2}(y^2-3x^2-1)\,y$$

and $e^{-x^2-y^2}$ is never zero, so $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ becomes

$$2x(3-3x^2+y^2) = 0 \quad \cdots (1)$$
$$2y(y^2-3x^2-1) = 0 \quad \cdots (2)$$

From (1), either $x = 0$ or $3x^2 - y^2 = 3$

Case I: $x = 0$. Then (2) becomes $2y(y^2-1) = 0$

so crit. points are $(0,0)$, $(0,1)$, $(0,-1)$

Case II: $3-3x^2+y^2 = 0$. Then (2) becomes $2y.(-4) = 0$

$\Rightarrow y = 0$, so $3 - 3x^2 = 0$ and critical points

are $(-1,0)$, $(0,0)$ and $(1,0)$.

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Now find boundary extrema, by solving

$$\left.\begin{array}{ll} \text{max/min} & f(x,y) \\ \text{subject to} & x^2+y^2=16 \end{array}\right\}$$

Put $g(x,y) = x^2+y^2$. Since we know $x^2+y^2=16$, this is the same as

$$\begin{array}{ll} \text{max/min} & e^{-16}(3x^2-y^2) \\ \text{subject to} & x^2+y^2=16 \end{array}$$

Put $h(x,y) = e^{-16}(3x^2-y^2)$ and solve $\begin{cases} \nabla h = \lambda \nabla g \\ g(x,y)=16 \end{cases}$

~~This is~~

$$6e^{-16}x = 2\lambda x \quad \ldots (1)$$
$$-2e^{-16}y = 2\lambda y \quad \ldots (2)$$
$$x^2+y^2 = 16 \quad \ldots (3)$$

If neither x nor $y=0$ then (1) and (2) give $\lambda = 3e^{-16} = -e^{-16}$ which is inconsistent. So either $x=0$ or $y=0$, and (3) shows that boundary extrema can only occur at $(4,0)$, $(-4,0)$, $(0,4)$ and $(0,-4)$.

$f(4,0) = 48e^{-16}$ $\quad f(-4,0) = 48e^{-16}$ $\quad f(0,4) = f(0,-4) = -16e^{-16}$

$f(0,0) = 0$ $\quad f(1,0) = 3e^{-1}$ $\quad f(-1,0) = 3e^{-1}$ $\quad f(0,1) = f(0,-1) = -e^{-1}$

So max value of f on the disc is $\underline{\underline{3e^{-1}}}$

min. value of f on the disc is $\underline{\underline{-e^{-1}}}$