SOLUTIONS : LIMITS AND CONTINUITY

SECTION ONE

(1) (a) This is $\lim_{t\to 0} f(t,0) = \lim_{t\to 0} 0 = 0$

(b) This is $\lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^2}{t^2+t^2} = \frac{1}{2}$

(c) The limit does not exist : if it did then the answers to (a) and (b) would necessarily be equal.

(2) (a) This is $\lim_{t\to 0} f(t,mt) = \lim_{t\to 0} \frac{3mt^3}{t^4+m^2t^2}$

$= \lim_{t\to 0} \frac{3mt}{t^2+m^2} = 0$

(b) This is $\lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{3t^4}{t^4+t^4} = 3/2$

(c) Since (a) and (b) are not equal, the limit cannot exist.

SECTION TWO

(1) (a) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{3r^3\cos^2\theta\sin\theta}{r}$

$= 0$

Since this is independent of θ, the limit exists (and equals zero)

(b) $$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\sin(\rho^2)}{\rho^2}$$

$$= 1$$

The limit exists and equals 1.

(c) $$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2+z^2}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^2\sin^2\phi(\cos^2\theta-\sin^2\theta)+\rho^2\cos^2\phi}{\rho^2}$$

$$= \sin^2\phi(\cos 2\theta)+\cos^2\phi$$

Since this depends on θ and ϕ, the limit does not exist.

SECTION THREE

1(a) Approaching $(0,0)$ along the x-axis gives:

$$\lim_{t\to 0} f(t,0) = 0$$

Approaching along the line $y=\sqrt{x}$ gives

$$\lim_{t\to 0} f(t,\sqrt{t}) = \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \frac{1}{2}$$

So the limit does not exist.

(b) The function $(x,y,z) \mapsto e^{x^2+y^2+z^2}$ is continuous and never zero, so the function $(x,y,z) \mapsto \frac{1}{e^{x^2+y^2+z^2}}$ is continuous (composition of continuous functions is continuous, and $x \mapsto 1/x$ is discontinuous

only at $x=0$). And $(x,y,z) \mapsto x+y+z$ is continuous, and the product of continuous functions is continuous, so

$$f(x,y,z) = \frac{x+y+z}{e^{x^2+y^2+z^2}}$$

is continuous.