SOLUTIONS : PARTIAL DERIVATIVES

1 (a) $f_x = \cos x \cos y$

$f_y = -\sin x \sin y$

(b) $f_x = \dfrac{1}{x+2y+3z}$

$f_y = \dfrac{2}{x+2y+3z}$

$f_z = \dfrac{3}{x+2y+3z}$

(c) $f_x = \frac{1}{y} \cos(x/y)\, e^{\sin(x/y)}$

$f_y = -\frac{x}{y^2} \cos(x/y)\, e^{\sin(x/y)}$

SECTION TWO

1 $f(x,y) = \frac{1}{2} \ln(x^2+y^2)$

so $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right)$

$= -\dfrac{2xy}{(x^2+y^2)^2}$

and $f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right)$

$= -\dfrac{2xy}{(x^2+y^2)^2}$

f_{xy} and f_{yx} are continuous wherever they are defined (i.e. away from the origin) so Clairaut's Theorem holds.

2. If such a function $f(x,y)$ existed then Clairaut's Theorem would say that $f_{xy} = f_{yx}$. But:

$$f_{xy} = \frac{\partial}{\partial y}(x+4y) = 4$$

$$f_{yx} = \frac{\partial}{\partial x}(3x-y) = 3$$

these are continuous functions

a contradiction! So no such function $f(x,y)$ exists

SECTION THREE

1 (a) Consult figure 1 on p 769 of the textbook. A vector in the direction of T_1 is $\langle 1, 0, \frac{\partial f}{\partial x} \rangle$, and a vector in the direction of T_2 is $\langle 0, 1, \frac{\partial f}{\partial y} \rangle$.

The tangent plane contains both T_1 and T_2, so is perpendicular to

$$\langle 1, 0, \frac{\partial f}{\partial x} \rangle \times \langle 0, 1, \frac{\partial f}{\partial y} \rangle = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$$

so the plane has equation

$$-\frac{\partial f}{\partial x} X - \frac{\partial f}{\partial y} Y + Z = \text{const.}$$

i.e. $$Z = \text{const.} + \frac{\partial f}{\partial x} X + \frac{\partial f}{\partial y} Y$$

(b) To compute the rate of change of $f(x,y)$ in the direction $\langle \alpha, \beta \rangle$ we can replace the graph of $z = f(x,y)$ by the graph $z = \text{const.} + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y$

or in other words replace f by its linear approximation

$$g(X,Y) = \text{const.} + \frac{\partial f}{\partial x} X + \frac{\partial f}{\partial y} Y$$

A path away from (a,b) in the direction $\langle \alpha, \beta \rangle$ is $X = a + \alpha t$, $Y = b + \beta t$

and the rate of change of $g(X,Y)$ in the direction $\langle \alpha, \beta \rangle$ is

$$\frac{d}{dt} g(X(t), Y(t)) \Big|_{t=0} = \frac{d}{dt}\left(\text{const.} + \frac{\partial f}{\partial x}(a + \alpha t) + \frac{\partial f}{\partial y}(b + \beta t) \right)\Bigg|_{t=0}$$

$$= \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}$$

Thus the rate of change of $f(x,y)$ in the direction of $\langle \alpha, \beta \rangle$ is $\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}$.

(c) If $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ then this is $(\nabla f) \cdot \langle \alpha, \beta \rangle$.