SECTION ONE

(1) The partial derivatives are

$$f_x = \frac{1}{1+(x+2y)^2}$$

$$f_y = \frac{2}{1+(x+2y)^2}$$

which are both continuous near (1,0) [they are rational functions and the denominator does not vanish]

Thus f is differentiable at $(1,0)$.

The linearization of f at $(1,0)$ is

$$f(1,0) + f_x(1,0)\,(x-1) + f_y(1,0)\,y$$

$$= \pi/4 + \tfrac{1}{2}(x-1) + y$$

so $$f(1.1, -0.05) \approx \pi/4 + \tfrac{1}{2}(1.1-1) + (-0.05)$$

$$= \underline{\underline{\pi/4}}$$

SECTION TWO

(1) $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (\cos x \cos y).\pi + (-\sin x \sin y).\tfrac{1}{2}t^{-1/2}$$

when $t=1$, $x=\pi$ and $y=1$ so

$$\frac{dz}{dt} = -\pi\cos 1$$

(2) $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{1}{y}\cdot e^{-t} + \left(-\frac{x}{y^2}\right)e^{-t}$$

$$= \frac{e^{-t}}{1+se^{-t}} - \frac{se^{-2t}}{(1+se^{-t})^2}$$

$$= \frac{1}{(1+se^{-t})^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}\cdot(-se^{-t}) - \frac{x}{y^2}(-se^{-t})$$

$$= \frac{-s}{(1+se^{-t})^2}$$

(3) Put r = radius

h = height

V = volume

so $\frac{dr}{dt} = 1.8$, $\frac{dh}{dt} = -2.5$, $V = \frac{1}{3}\pi r^2 h$

Then
$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$
$$= \tfrac{2}{3}\pi r h\,(1.8) + \tfrac{1}{3}\pi r^2(-2.5)$$
$$\approx \underline{43731 \text{ in}^3/\text{s}}$$

so the volume is increasing at a rate of approximately 43731 in³/second.

SECTION THREE

(1) (a) Put $u = x + ct$

$v = x - ct$

Then
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}$$
$$= f'(u).\,c + g'(v).\,(-c)$$

so $$\frac{\partial z}{\partial t} = c f'(x+ct) - c\, g'(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial t}\right)\cdot\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial t}\right)\cdot\frac{\partial v}{\partial t}$$

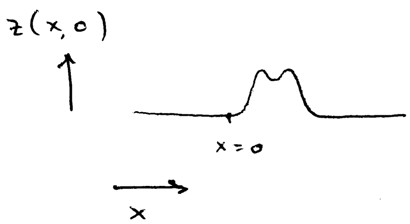
$$= c f''(u)\,.\,c - c\, g''(v)\,.\,(-c)$$

$$= c^2 f''(x+ct) + c^2 g''(x-ct)$$

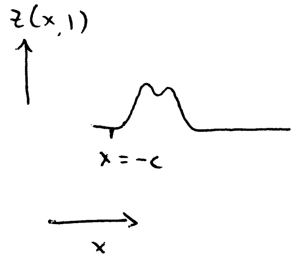
Similarly $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$$

and so z is a solution to $z_{tt} = c^2 z_{xx}$.

(b) Consider $z = f(x+ct)$. If it looks like this:

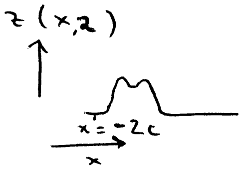


at $t=0$ then it will look like this:



at $t=1$

and like this:



at $t=2$.

In other words, $z = f(x-ct)$ represents a wave in the shape of the graph of f which moves leftwards along the string with speed c. Similarly, $z = g(x-ct)$ represents a wave of shape the graph of g moving rightwards along the string at speed c.

(c) A computation very similar to that in part (a) gives

$$z_{tt} = c^2(z_{uu} - 2z_{uv} + z_{vv})$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}$$

So the wave equation $z_{tt} = c^2 z_{xx}$ can be rewritten as

$$c^2(z_{uu} - 2z_{uv} + z_{vv}) = c^2(z_{uu} + 2z_{uv} + z_{vv})$$

or in other words $z_{uv} = 0$.

Integrating with respect to v: $\frac{\partial z}{\partial u} = F(u)$

and integrating with respect to u: $z = f(u) + g(v)$

where f is the antiderivative of F. Done.