SOLUTIONS: CHAIN RULE / DIRECTIONAL DERIVATIVES

SECTION ONE

(1) $x = r\cos\theta$ and $y = r\sin\theta$

so $$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}$$

and $$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}$$

When $f(x,y) = \sqrt{x^2+y^2}$ we find

$$\frac{\partial f}{\partial r} = \cos\theta\cdot\frac{x}{\sqrt{x^2+y^2}} + \sin\theta\cdot\frac{y}{\sqrt{x^2+y^2}}$$

$$= \cos\theta\cdot\frac{r\cos\theta}{r} + \sin\theta\cdot\frac{r\sin\theta}{r}$$

$$= 1$$

$$\frac{\partial f}{\partial \theta} = -r\sin\theta\cdot\frac{x}{\sqrt{x^2+y^2}} + r\cos\theta\,\frac{y}{\sqrt{x^2+y^2}}$$

$$= -r\sin\theta\cos\theta + r\cos\theta\sin\theta = 0$$

This is what we expect, as $f(x,y) = r$.

(2) $$\frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt}$$

when $t = 3$, $x = 2$ and $y = 3$

so $\frac{\partial T}{\partial x} = 4$ and $\frac{\partial T}{\partial y} = 3$

also $$\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}} = 1/4$$

and $$\frac{dy}{dt} = 1/3$$

so $$\frac{dT}{dt} = 4 \cdot 1/4 + 3 \cdot 1/3 = 2$$

The bug's temperature is rising at 2 °C per second.

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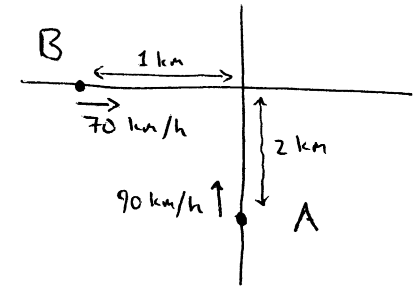
(3)



NOT TO SCALE

Let the distance from A to the intersection be a and the distance from B to the intersection be b.

Let the distance from A to B be D.

Then $D = \sqrt{a^2+b^2}$

and

$$\frac{dD}{dt} = \frac{\partial D}{\partial a}\cdot\frac{da}{dt} + \frac{\partial D}{\partial b}\cdot\frac{db}{dt}$$

$$= \frac{a}{\sqrt{a^2+b^2}}\cdot\frac{da}{dt} + \frac{b}{\sqrt{a^2+b^2}}\cdot\frac{db}{dt}$$

$$= \frac{2}{\sqrt{5}}\cdot(-90) + \frac{1}{\sqrt{5}}\cdot(-70)$$

$$= \frac{-250}{\sqrt{5}} \approx -112$$

The distance from A to B is decreasing at approximately 112 km/h.

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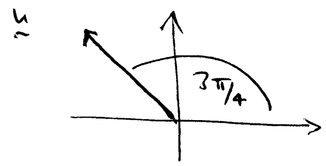
SECTION TWO

(1) $\qquad \underset{\sim}{\nabla} f = \langle \cos(x+2y), 2\cos(x+2y) \rangle$

so $\quad \underset{\sim}{\nabla} f(4,2) = \langle \cos 8, 2\cos 8 \rangle$



Direction vector $\underset{\sim}{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$

so the directional derivative

is $\quad \langle \cos 8, 2\cos 8 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = \dfrac{\cos 8}{\sqrt{2}}$

The direction of steepest increase is that of ∇f, i.e. $\langle \cos 8, 2\cos 8 \rangle$

i.e. the direction $\left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$

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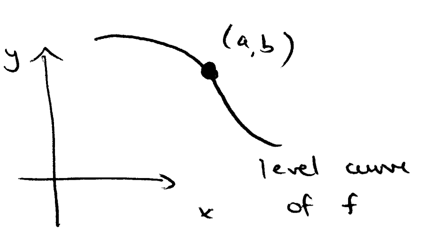
SECTION THREE

(1)



Let $(x(t), y(t))$ be the co-ordinates of a point moving along the level curve which is at (a,b) when $t=0$.

Then $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \Big|_{t=0}$ ~~points~~ is tangent to the level curve. But

$$f(x(t), y(t)) = \text{constant}.$$

(since we're moving along a level curve)

so $\frac{d}{dt}\left(f(x(t), y(t))\right) = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$$

at $t=0$, this is ∇f at (a,b)

at $t=0$, this points along the tangent line to the level curve

Done.