SOLUTIONS : DIRECTIONAL DERIVATIVES / GRADIENT VECTOR

SECTION ONE :

(1) $\nabla f = \langle -e^y \sin x, \; e^y \cos x \rangle$

so $\nabla f(\pi/4, 0) = \langle -1/\sqrt{2}, \; 1/\sqrt{2} \rangle$

Direction vector $\hat{u} = \frac{1}{\sqrt{10}} \underline{i} + \frac{3}{\sqrt{10}} \underline{j}$

so $D_u f = \nabla f \cdot \hat{u} = \underline{\underline{2/\sqrt{10}}}$

(2) $\nabla f = \langle y^2 z^3, \; 2xyz^3, \; 3xy^2z^2 \rangle$

So $\nabla f(3, 2, 1) = \langle 4, 12, 36 \rangle$

Direction vector $\hat{u} = \langle \frac{1}{\sqrt{3}}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

so $D_{\hat{u}} f = \nabla f \cdot \hat{u} = \underline{\underline{\frac{52}{\sqrt{3}}}}$

SECTION TWO

(1) $\nabla f = \langle 1, 1/z, -y/z^2 \rangle$, so $\nabla f(4, 3, -1) = \langle 1, -1, -3 \rangle$

The maximum rate of change is $\|\nabla f\| = \sqrt{11}$

and it occurs in the direction $\langle \frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}} \rangle$.

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(2) We know $T(x,y,z) = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$

and since $T(1,2,2) = 120$ we know that the constant $k = 1080$.

(a) $$\underset{\sim}{\nabla} T = \left\langle \frac{-kx}{(x^2+y^2+z^2)^{3/2}}, \frac{-ky}{(x^2+y^2+z^2)^{3/2}}, \frac{-kz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\underset{\sim}{\nabla} T(1,2,2) = \langle -40, -80, -80 \rangle.$$

Direction is $\dfrac{\langle 1,-1,1\rangle}{\sqrt{3}} = \langle \frac{1}{\sqrt{3}}, -1/\sqrt{3}, 1/\sqrt{3} \rangle$

So directional derivative $= \dfrac{-40}{\sqrt{3}}$

(b) Direction of steepest increase of temperature $= \underset{\sim}{\nabla} T$

$$= \frac{k}{(x^2+y^2+z^2)^{3/2}} \langle -x, -y, -z \rangle$$

This is proportional to $\langle -x, -y, -z \rangle$, so points towards the origin, which is the center of the ball.

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SECTION THREE

(1) Put $f(x,y,z) = \frac{x^2}{4} + y^2 + z^2/9$

Normal vector to tangent plane is $\underset{\sim}{\nabla} f$

$$\underset{\sim}{\nabla} f \Big|_{(2,0,3)} = \left\langle x/2,\ 2y,\ 2z/9 \right\rangle \Big|_{(2,0,3)}$$

$$= \langle 1, 0, 2/3 \rangle$$

So the tangent plane is

$$x + 2z/3 = \text{const.}$$

Since $(2,0,3)$ is on the plane, it is

$$\underline{\underline{x + 2z/3 = 4}}$$

(2) Put $g(x,y,z) = z - f(x,y)$.

Normal vector to tangent plane is

$$\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

so the tangent plane to $z = f(x,y)$ at $(a,b,f(a,b))$ is

$$-f_x(a,b)\,x - f_y(a,b)\,y + z = f(a,b) - af_x(a,b) - bf_y(a,b)$$

or in other words

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(3) Put $f(x,y,z) = x^2+y^2-z^2$

$g(x,y,z) = x^2+y^2+z$

Tangent plane to hyperboloid has normal vector

$$\underset{\sim}{\nabla} f = \langle 2x, 2y, -2z \rangle$$

which at $(1,1,-1)$ is $\langle 2,2,2 \rangle$.

Thus the tangent plane is

$$2x+2y+2z = 2$$

or

$$x+y+z = 1.$$

The normal line to the paraboloid

$$g(x,y,z) = 4$$

at $(2,2,-4)$ has direction vector $\nabla g \big|_{(2,2,-4)}$

$$\nabla g \big|_{(2,2,-4)} = \langle 2x, 2y, 1 \rangle \big|_{(2,2,-4)}$$

$$= \langle 4, 4, 1 \rangle$$

so the line is

$$\langle 2, 2, -4 \rangle + t \langle 4, 4, 1 \rangle$$

$t \in \mathbb{R}$ is a parameter

They meet when

$$(2+4t) + (2+4t) + (t-4) = 1$$

$$\Rightarrow \quad 9t = 1$$

so they meet at $\langle 2 + 4/9,\ 2 + 4/9,\ 1/9 - 4 \rangle$

$$= \underline{\underline{\langle 22/9,\ 22/9,\ -35/9 \rangle}}$$