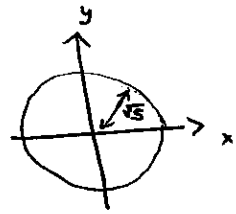
SURFACES : SOLUTIONS

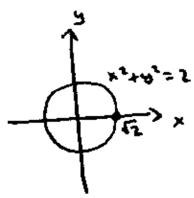
SECTION 1

1 (a) $x^2+y^2-z^2=1$, so the trace on the horizontal plane $z=-2$ is $x^2+y^2=5$

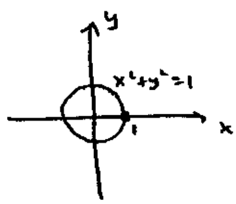


trace at $k=-2$

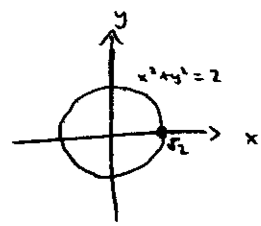
Similarly, the traces in the planes $z=-1, 0, 1$ and 2 are



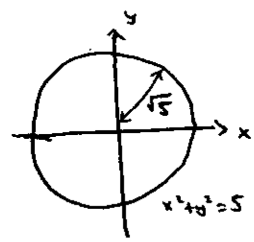
$z=-1$



$z=0$

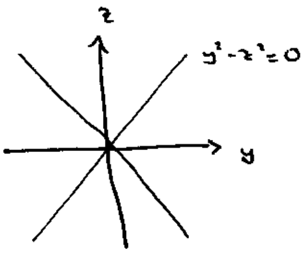


$z=1$

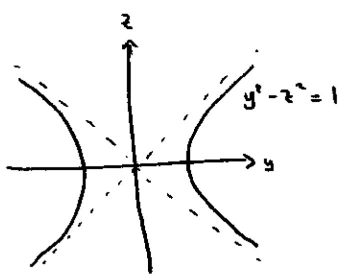


$z=2$

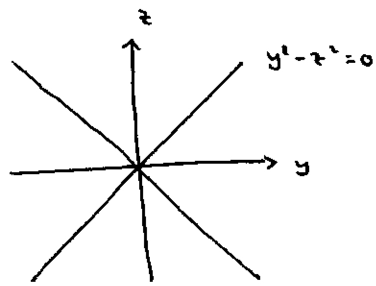
(b) Traces in the horizontal planes $x=k$ are



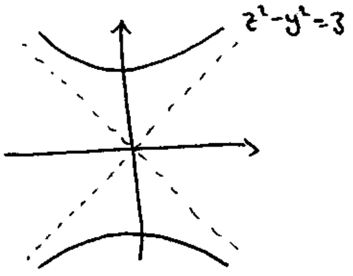
$x=-1$



$x=0$

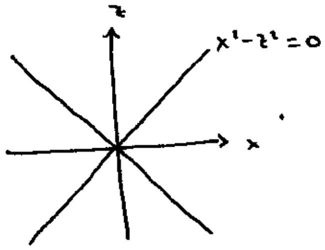


$x=1$

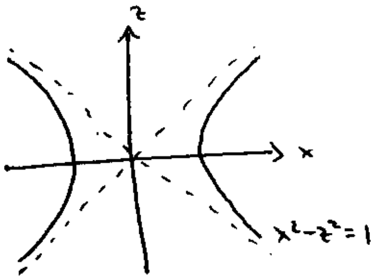


$x=2$

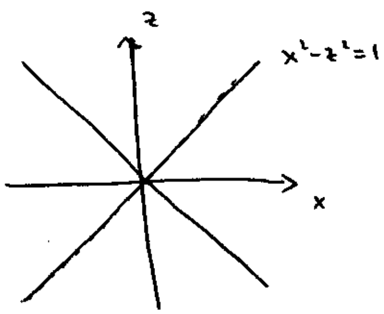
(c) The traces in the vertical planes $y=k$ are



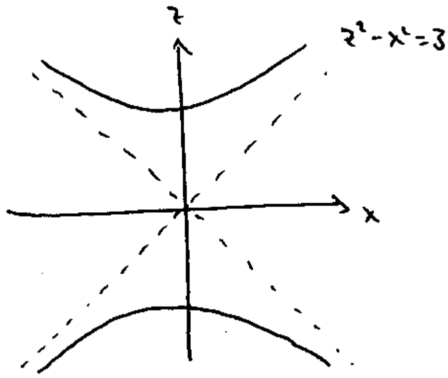
$y=-1$



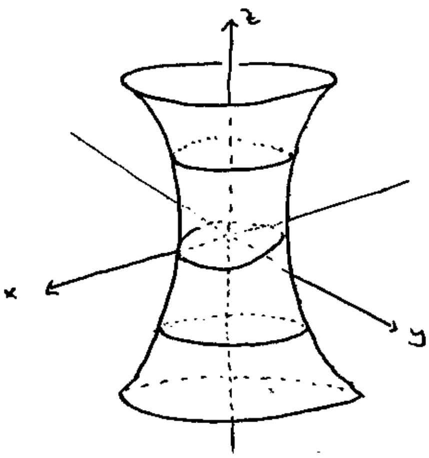
$y=0$



$y=1$



(d) The surface looks like



SURFACES : SOLUTIONS

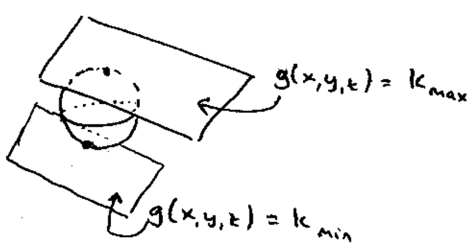
SECTION TWO

Here is a sketch solution to the "More Challenging Problem"

The level surfaces of $f(x,y,z) = 3x+4y+12z$ are parallel planes with normal vector $\langle 3,4,12\rangle$.

The level surfaces of $f(x,y,z)$ are spheres centred at the origin.

The maximum value k of $g(x,y,z) = 3x+4y+12z$ on the sphere $S = \{(x,y,z) : x^2+y^2+z^2=1\}$ occurs when the level surface $g(x,y,z)=k$ is tangent to the sphere S.



The normal vector to the tangent plane to S at the point (x,y,z) is $\langle x,y,z\rangle$ so we need $\langle x,y,z\rangle$ to be parallel to $\langle 3,4,12\rangle$ (which is the normal vector to the plane) Solving:

$$\langle x,y,z\rangle = \lambda\langle 3,4,12\rangle \quad \text{and} \quad x^2+y^2+z^2=1$$

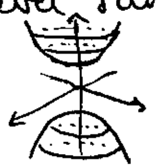
gives $(x,y,z) = \left(\tfrac{3}{13}, \tfrac{4}{13}, \tfrac{12}{13}\right)$ max $\leftarrow g=13$

or $\left(-\tfrac{3}{13}, -\tfrac{4}{13}, -\tfrac{12}{12}\right)$ min $\leftarrow g=-13$

SURFACES : SOLUTIONS

$h(x,y,z) = 4z^2 - x^2 - 2y^2$ is minimized / maximized on S where the level surfaces for h are tangent to S.

For $k > 0$, the level surfaces $h(x,y,z) = k$ are hyperboloids of two sheets



and the closest point to the origin on such a surface is $(0, 0, \frac{1}{2}\sqrt{k})$ or $(0, 0, -\frac{1}{2}\sqrt{k})$

Thus any tangency to S must occur at these points, which are on the z-axis. Since they are also on S, they are the points $(0,0,\pm 1)$ and $h = 4$ there.

For $k < 0$ the level surfaces are hyperboloids of one sheet, and a similar argument shows that there are tangencies between S and level surfaces of h at $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$.

$h(\pm 1, 0, 0) = -1$ and $h(0, \pm 1, 0) = -2$, so the minimum value of h on S is -2 and the max. is 4.

Matching games: p110 1 = III 2 = VI 3 = V 4 = I 5 = IV 6 = II

p111 1 = VI 2 = I 3 = II 4 = IV 5 = V 6 = III