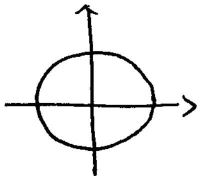
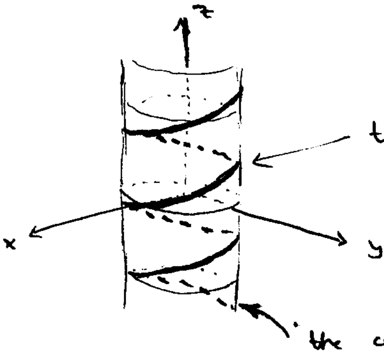
SOLUTIONS : SPACE CURVES

1 (a)



The unit circle

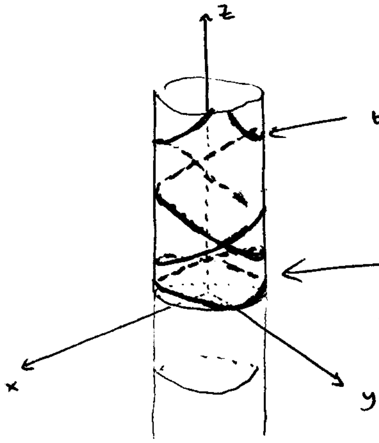
(b)



the curve

the cylinder $x^2+y^2=1$, which the curve lies on

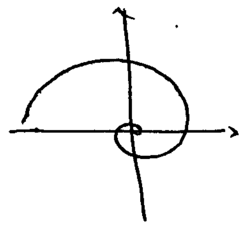
(2)



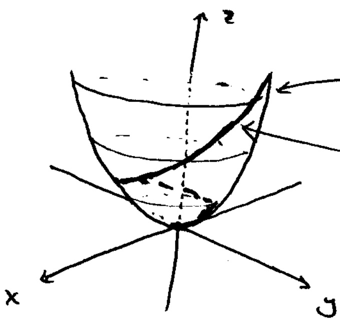
the cylinder $x^2+y^2=1$, on which the curve lies

the curve. Don't forget that t can be negative!

(3) Projecting to the xy plane we see a spiral



The curve lies on $z = x^2+y^2$, which is a paraboloid, so:



the paraboloid $z = x^2+y^2$

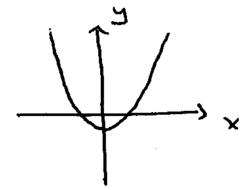
the curve

SOLUTIONS : SPACE CURVES

SECTION TWO

1 (a) Project to the xy plane (i.e. eliminate z) to find

$$x^2+y^2 = (1+y)^2 \quad \Rightarrow \quad x^2 = 2y+1$$
$$\Rightarrow \quad y = \tfrac{1}{2}(x^2-1)$$



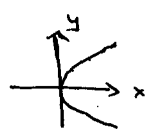
Parametrize this by setting $x = t$, $y = \frac{1}{2}(t^2-1)$, $\quad -\infty < t < \infty$

Then $z = 1+y = \frac{1}{2}(t^2+1)$, so

$$\underline{r}(t) = \left\langle t, \tfrac{1}{2}(t^2-1), \tfrac{1}{2}(t^2+1) \right\rangle \qquad t \in \mathbb{R}$$

(b) Projecting to the xy plane (eliminating z) we find

$x = y^2$. Parametrize this as $x = t^2$, $y = t$ $\quad -\infty < t < \infty$



Since $z = x^2 + 3y^2$, we get the parametrization

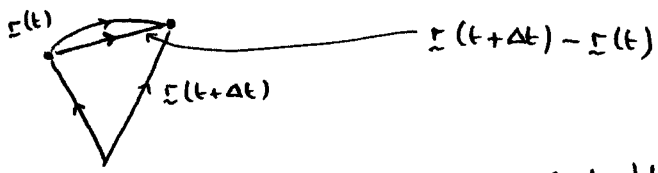
$$\underline{r}(t) = \langle t^2, t, t^4+3t^2 \rangle \qquad t \in \mathbb{R}$$

for the curve.

SECTION THREE

Imagine chopping up the time interval $a \le t \le b$ into N very small intervals of Δt (so $N = \frac{b-a}{\Delta t}$)

In one such interval, a particle moving along the curve (whose position vector at time t is $\underline{r}(t)$) moves from position $\underset{\sim}{r}(t)$ to $\underset{\sim}{r}(t+\Delta t)$. Thus the distance it moves is approximately



$$\| \underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t) \|$$

and this is approximately

$$\| \underset{\sim}{r}'(t)\, \Delta t \|$$

(look at the definition of $\underset{\sim}{r}'(t)$ as a limit!)

So
$$\text{length of } C \approx \sum_{N \text{ intervals}} \| \underset{\sim}{r}'(t_i)\, \Delta t \|$$

where t_i is the beginning of the i^{th} interval, $t_i = a + i\frac{(b-a)}{N} = a + i\Delta t$

Letting $\Delta t \to 0$, this approximation gets better and better, and the sum tends to the integral $\int_a^b \| \underset{\sim}{r}'(t) \| \, dt$

So
$$L = \int_a^b \| \underset{\sim}{r}'(t) \| \, dt$$

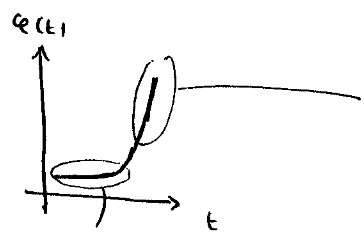
Suppose $\underset{\sim}{r}_1(s)$ is a parametrization of C, and $\underset{\sim}{r}_2(t)$ is another parametrization. We can think of them as particles moving along C at different speeds. To turn the first parametrization into the second, we need to get the first particle to move along C faster or slower (so it matches the second particle). One way to do this is to let time, for the first particle, run faster (to make the particle faster) or slower (to make the particle move slower).

At time t, the second particle will be at some point P.

Let $\varphi(t)$ be the time at which the <u>first</u> particle is at P. This function φ is what keeps track of the speeding up or slowing down of time for the first particle to make it match the second particle:

$$\underset{\sim}{r}_1(\varphi(t)) = \underset{\sim}{r}_2(t) \qquad \text{for all } t$$

<u>example</u>



time is being sped up here as the slope of the line is large

time is being slowed down here, as the slope of the line is small

But now:

$$L = \int_{a_2}^{b_2} \| r_2'(t) \| \, dt$$

$$= \int_{a_2}^{b_2} \| r_1(\varphi(t))' \| \, dt$$

$$= \int_{a_2}^{b_2} \| r_1'(\varphi(t)) \| \, \varphi'(t) \, dt \qquad \text{chain rule}$$

$$= \int_{a_1}^{b_1} \| r_1'(s) \| \, ds \qquad \text{putting } s = \varphi(t). \quad \text{Done.}$$