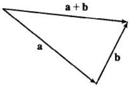
◆ EXERCISES ◆

1. (a) The radius of the sphere is the distance between the points $(-1, 2, 1)$ and $(6, -2, 3)$, namely $\sqrt{[6-(-1)]^2 + (-2-2)^2 + (3-1)^2} = \sqrt{69}$. By the formula for an equation of a sphere (see page 650), an equation of the sphere with center $(-1, 2, 1)$ and radius $\sqrt{69}$ is $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$.

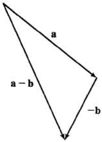
(b) The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $(y-2)^2 + (z-1)^2 = 68$, $x = 0$ which represents a circle in the yz-plane with center $(0, 2, 1)$ and radius $\sqrt{68}$.

(c) Completing squares gives $(x-4)^2 + (y+1)^2 + (z+3)^2 = -1 + 16 + 1 + 9 = 25$. Thus, the sphere is centered at $(4, -1, -3)$ and has radius 5.

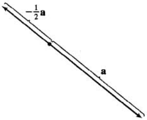
2. (a)



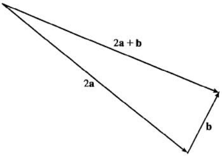
(b)



(c)



(d)



3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

4. (a) $2\mathbf{a} + 3\mathbf{b} = 2\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} + 9\,\mathbf{i} - 6\,\mathbf{j} + 3\,\mathbf{k} = 11\,\mathbf{i} - 4\,\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

(c) $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\,\mathbf{i} - 7\,\mathbf{j} - 5\,\mathbf{k}$

(e) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}$, $|\mathbf{b} \times \mathbf{c}| = 3\sqrt{9 + 25 + 1} = 3\sqrt{35}$

(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\,\mathbf{i}+15\,\mathbf{j}+3\,\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = (3+30)\,\mathbf{i} - (3+18)\,\mathbf{j} + (15-9)\,\mathbf{k}$$

$$= 33\,\mathbf{i} - 21\,\mathbf{j} + 6\,\mathbf{k}.$$

(i) The scalar projection is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\frac{1}{\sqrt{6}}(\mathbf{a}/|\mathbf{a}|) = -\frac{1}{6}(\mathbf{i}+\mathbf{j}-2\,\mathbf{k})$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\,\sqrt{14}} = \dfrac{-1}{2\sqrt{21}}$ and $\theta = \cos^{-1}\dfrac{-1}{2\sqrt{21}} \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3,2,x\rangle\cdot\langle 2x,4,x\rangle = 0 \;\Leftrightarrow\; (3)(2x)+(2)(4)+(x)(x) = 0 \;\Leftrightarrow\; x^2+6x+8=0 \;\Leftrightarrow\; (x+2)(x+4)=0 \;\Leftrightarrow\; x=-2$ or $x=-4$.

6. We know that the cross product of two vectors is orthogonal to both. So we calculate $(\mathbf{j}+2\,\mathbf{k})\times(\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k}) = [3-(-4)]\,\mathbf{i} - (0-2)\,\mathbf{j} + (0-1)\,\mathbf{k} = 7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}$. Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}}{\sqrt{7^2+2^2+(-1)^2}} = \pm\dfrac{1}{3\sqrt{6}}(7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k})$, that is, $\frac{7}{3\sqrt{6}}\,\mathbf{i}+\frac{2}{3\sqrt{6}}\,\mathbf{j}-\frac{1}{3\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\,\mathbf{i}-\frac{2}{3\sqrt{6}}\,\mathbf{j}+\frac{1}{3\sqrt{6}}\,\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8. $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a})$

(see Exercise 9.4.30)

$= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

$= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0,0,0)$ to $(1,1,1)$ and $(1,0,0)$ to $(0,1,1)$ are $\langle 1,1,1\rangle$ and $\langle -1,1,1\rangle$. Let θ be the angle between these two vectors. $\langle 1,1,1\rangle\cdot\langle -1,1,1\rangle = -1+1+1 = 1 = |\langle 1,1,1\rangle|\,|\langle -1,1,1\rangle|\cos\theta = 3\cos\theta$ $\Rightarrow\; \cos\theta = \frac{1}{3} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1, 3, -1 \rangle$, $\overrightarrow{AC} = \langle -2, 1, 3 \rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1 \rangle$. By Equation 9.4.7,

$$\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$ The volume is

$\left|\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$, $W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ joules

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ (1), and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \Rightarrow$ $F_1 = F_2 \dfrac{\sin 30^\circ}{\sin 20^\circ}$ (2). Substituting (2) into (1) gives $F_2(\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \Rightarrow F_2 \approx 114$ N. Substituting this into (2) gives $F_1 \approx 166$ N.

14. $|\tau| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50)\sin(90^\circ - 30^\circ) \approx 17.3$ joules

15. $x = 1 + 2t$, $y = 2 - t$, $z = 4 + 3t$

16. $\mathbf{v} = \langle 8, -2, 5 \rangle$, so $x = -6 + 8t$, $y = -1 - 2t$ and $z = 5t$.

17. $\mathbf{v} = \langle 4, -3, 5 \rangle$, so $x = 1 + 4t$, $y = -3t$, $z = 1 + 5t$.

18. $2(x - 4) + 6(y + 1) - 3(z + 1) = 0$ or $2x + 6y - 3z = 5$.

19. Since the two planes are parallel, they will have the same normal vectors. So se can take $\mathbf{n} = \langle 1, 2, 5 \rangle$ and an equation of the plane is $1\,[x - (-4)] + 2(y - 1) + 5(z - 2) = 0$ or $x + 2y + 5z = 8$.

20. Here the vectors $\mathbf{a} = \langle 2 - (-1), 0 - 2, 1 - 0 \rangle = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle -5 - (-1), 3 - 2, 1 - 0 \rangle = \langle -4, 1, 1 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -3, -7, -5 \rangle$ is a normal vector to the plane and an equation of the plane is $-3\,[x - (-1)] - 7\,(y - 2) - 5\,(z - 0) = 0$ or $3x + 7y + 5z = 11$.

21. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, the normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = 3\,\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \Leftrightarrow x + y + z = 4$.

22. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2 - t) - (1 + 3t) + 4t = 2 \Rightarrow -t + 3 = 2 \Rightarrow t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

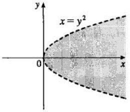
(b) $\cos\theta = \dfrac{\langle 1,1,-1\rangle \cdot \langle 2,-3,4\rangle}{\sqrt{3}\,\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ (or we can say $\approx 58^\circ$).

25. By Exercise 9.5.49, $D = \dfrac{|2-24|}{\sqrt{26}} = \dfrac{22}{\sqrt{26}}$.

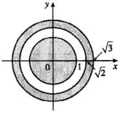
26. Use the formula proven in Exercise 9.4.27. In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1,-1,2\rangle$. A point on the line is $(1,2,-1)$ (setting $t=0$), and therefore $\mathbf{b} = \langle 1-0, 2-0, -1-0\rangle = \langle 1,2,-1\rangle$. Hence

$$d = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1,-1,2\rangle \times \langle 1,2,-1\rangle|}{\sqrt{1+1+4}} = \frac{|\langle -3,3,3\rangle|}{\sqrt{6}} = \sqrt{\frac{27}{6}} = \frac{3}{\sqrt{2}}.$$

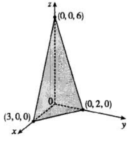
27. $\ln(x-y^2)$ is defined only when $x - y^2 > 0$, or $x > y^2$, and x is defined for all real numbers, so the domain of the product $x\ln(x-y^2)$ is $\{(x,y) \mid x > y^2\}$.



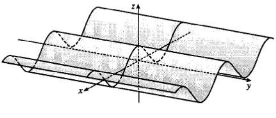
28. We need $\sin\pi(x^2+y^2) \geq 0 \;\Leftrightarrow\; 2n\pi \leq \pi(x^2+y^2) \leq (2n+1)\pi$, n an integer, so $D = \{(x,y) \mid 2n \leq x^2+y^2 \leq 2n+1, n \text{ an integer}\}$.



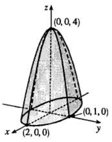
29. The graph is the plane $z = 6 - 2x - 3y \;\Rightarrow\; 2x + 3y + z = 6$. The intercepts with the coordinate axes are $(3,0,0)$, $(0,2,0)$, and $(0,0,6)$ which enable us to sketch the portion of the plane that lies in the first octant.



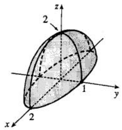
30. The equation is $z = \cos x$, which doesn't involve y. Thus the traces in $y = k$ are the graph $z = \cos x$, $y = k$, giving a cylindrical surface.



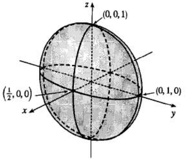
31. The equation is $z = 4 - x^2 - 4y^2$. The traces in $x = k$ are $z = 4 - k^2 - 4y^2$, a family of parabolas opening downward, as are the traces in $y = k$, $z = 4 - 4k^2 - x^2$. The traces in $z = k$ are $x^2 + 4y^2 = 4 - k$, a family of ellipses, so the surface is an elliptic paraboloid.



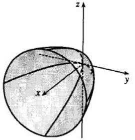
32. The equation is $z = \sqrt{4 - x^2 - 4y^2}$ or $x^2 + 4y^2 + z^2 = 4, z \geq 0$. The traces in $x = k$, $y = k$, and $z = k$ are ellipses or portions of ellipses, and the equation can be recognized as that of an ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$, $z \geq 0$, with intercepts ± 2, ± 1, and 2 for x, y, and z respectively. Since $z \geq 0$, we have only the upper half of the ellipsoid.



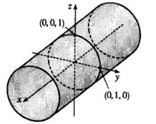
33. An equivalent equation is $\frac{x^2}{(1/2)^2} + y^2 + z^2 = 1$, an ellipsoid centered at the origin with intercepts $\pm\frac{1}{2}$, ± 1, and ± 1.



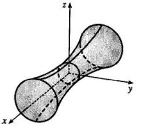
34. $x = y^2 + z^2$ is the equation of a circular paraboloid opening in the direction of the positive x-axis.



35. $y^2 + z^2 = 1$ is the equation of a circular cylinder with axis the x-axis.



36. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



37. $x = 2\cos\frac{\pi}{6} = \sqrt{3}$, $y = 2\sin\frac{\pi}{6} = 1$, $z = 2$, so in rectangular coordinates the point is $\left(\sqrt{3}, 1, 2\right)$. $\rho = \sqrt{3 + 1 + 4} = 2\sqrt{2}$, $\theta = \frac{\pi}{6}$, and $\cos\phi = z/\rho = \frac{1}{\sqrt{2}}$, so $\phi = \frac{\pi}{4}$ and the spherical coordinates are $\left(2\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4}\right)$.

38. $r = \sqrt{4 + 4} = 2\sqrt{2}$, $z = -1$, $\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$ and in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4 + 4 + 1} = 3$, $\cos\phi = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

39. $x = 4\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 1$, $y = 4\sin\frac{\pi}{6}\sin\frac{\pi}{3} = \sqrt{3}$, $z = 4\cos\frac{\pi}{6} = 2\sqrt{3}$ so in rectangular coordinates the point is $\left(1, \sqrt{3}, 2\sqrt{3}\right)$. $r^2 = x^2 + y^2 = 4$, $r = 2$, so the cylindrical coordinates are $\left(2, \frac{\pi}{3}, 2\sqrt{3}\right)$.

40. (a) $\theta = \frac{\pi}{4}$. In spherical coordinates, this is a half-plane including the z-axis and intersecting the xy-plane in the half-line $x = y$, $x > 0$.

(b) $\phi = \frac{\pi}{4}$. This is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

41. $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.

42. $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2\cos^2\phi = 4$ or $\rho^2\sin^2\phi = 4$ or $\rho\sin\phi = 2$.

43. The resulting surface is a circular paraboloid with equation $z = 4x^2 + 4y^2$. Changing to cylindrical coordinates we have $z = 4\left(x^2 + y^2\right) = 4r^2$.

44. $\rho = 2\cos\phi \;\Rightarrow\; \rho^2 = 2\rho\cos\phi \;\Rightarrow\; x^2 + y^2 + z^2 = 2z \;\Rightarrow$ $x^2 + y^2 + (z - 1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \le \rho \le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.