

into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \Rightarrow Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft³/s, and the remaining 1065 ft³/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.



Review

• CONCEPT CHECK •

- (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

(b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy -plane.
- A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.
- $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f . We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.
- (a) See Definition 11.2.3.

(b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.
- (a) See (2) and (3) in Section 11.3.

(b) See the discussion preceding Example 2 on page 769.

(c) To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x . To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .
- See the statement of Clairaut's Theorem on page 773.
- (a) See (2) in Section 11.4.

(b) See (19) and the preceding discussion in Section 11.6.

(c) See the discussion following Example 6 on page 787.
- See (3) and (4) and the accompanying discussion in Section 11.4. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b) . Thus it is the linear function which best approximates f near (a, b) .
- (a) See Definition 11.4.7.

(b) Use Theorem 11.4.8.
- See (10) and the associated discussion in Section 11.4.

11. See (2) and (3) in Section 11.5.
12. See (7) and the preceding discussion in Section 11.5.
13. (a) See Definition 11.6.2. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P . (See Figure 5 in Section 11.6.)
(b) See Theorem 11.6.3.
14. (a) See (8) and (13) in Section 11.6.
(b) $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
(c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.
15. (a) f has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) .
(b) f has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f .
(c) f has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) .
(d) f has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f .
(e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.
16. (a) By Theorem 11.7.2, if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
(b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.
17. See (3) in Section 11.7.
18. (a) See Figure 11 and the accompanying discussion in Section 11.7.
(b) See Theorem 11.7.8.
(c) See the procedure outlined in (9) in Section 11.7.
19. See the discussion beginning on page 822; see the discussion preceding Example 5 on page 826.

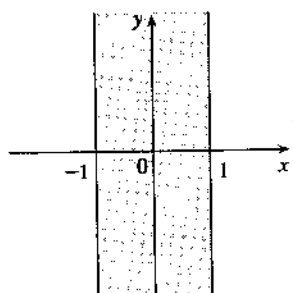
▲ TRUE-FALSE QUIZ ▲

1. True. $f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$ from Equation 11.3.3. Let $h = y - b$. As $h \rightarrow 0$, $y \rightarrow b$. Then by substituting, we get $f_y(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}$.
2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.
3. False. $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$.
4. True. From Equation 11.6.14 we get $D_{\mathbf{k}}f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.
5. False. See Example 11.2.3.
6. False. See Exercise 11.4.40(a).

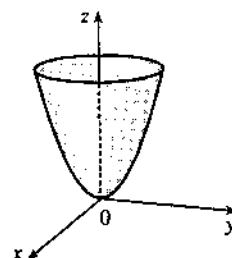
7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 11.7.2, $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \langle 0, 0 \rangle = \mathbf{0}$.
8. False. The limit does not exist because the function is not defined on the line $y = x$, and so we have a path approaching the point $(1, 1)$ along which f does not approach $\frac{1}{2}$.
9. False. $\nabla f(x, y) = \langle 0, 1/y \rangle$.
10. True. This is part (c) of the Second Derivatives Test (11.7.3).
11. True. $\nabla f = \langle \cos x, \cos y \rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos \theta| \leq 1$, so $|\nabla f| \leq \sqrt{2}$. Now $D_{\mathbf{u}}f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}f(x, y)| \leq \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.
12. False. See Exercise 11.7.29.

◆ EXERCISES ◆

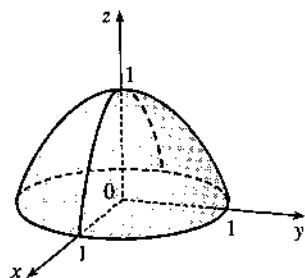
1. The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$ while the domain of $\tan^{-1} y$ is all real numbers, so the domain of $f(x, y) = \sin^{-1} x + \tan^{-1} y$ is $\{(x, y) \mid -1 \leq x \leq 1\}$.



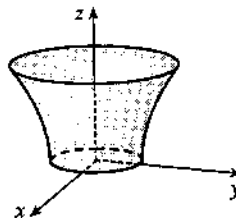
2. $D = \{(x, y, z) \mid z \geq x^2 + y^2\}$, the points on and above the paraboloid $z = x^2 + y^2$.



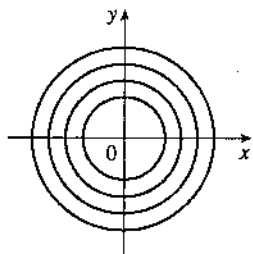
3. $z = f(x, y) = 1 - x^2 - y^2$, a paraboloid with vertex $(0, 0, 1)$.



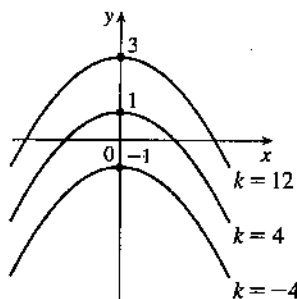
4. $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$, so $z \geq 0$ and $1 = x^2 + y^2 - z^2$. Thus the graph is the upper half of a hyperboloid of one sheet.



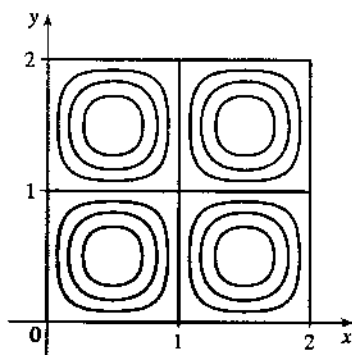
5. Let $k = e^{-c} = e^{-(x^2+y^2)}$ be the level curves. Then $-\ln k = c = x^2 + y^2$, so we have a family of concentric circles.



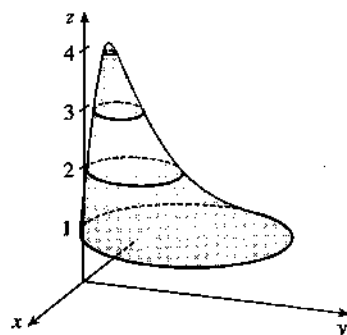
6. $k = x^2 + 4y$ or $4(y - k/4) = -x^2$, a family of parabolas with vertex at $(0, k/4)$.



7.



8.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\lim_{(x,y) \rightarrow (1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.
10. As $(x, y) \rightarrow (0, 0)$ along the x -axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ along this line. But $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x, y) \rightarrow (0, 0)$ along the line $x = y$, $f(x, y) \rightarrow \frac{2}{3}$. Thus the limit doesn't exist.
11. (a) $T_x(6, 4) = \lim_{h \rightarrow 0} \frac{T(6+h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6, 4) \approx \frac{T(8, 4) - T(6, 4)}{2} = \frac{86 - 80}{2} = 3$,
 $T_x(6, 4) \approx \frac{T(4, 4) - T(6, 4)}{-2} = \frac{72 - 80}{-2} = 4$. Averaging these values, we estimate $T_x(6, 4)$ to be approximately 3.5°C/m . Similarly, $T_y(6, 4) = \lim_{h \rightarrow 0} \frac{T(6, 4+h) - T(6, 4)}{h}$, which we can approximate with $h = \pm 2$: $T_y(6, 4) \approx \frac{T(6, 6) - T(6, 4)}{2} = \frac{75 - 80}{2} = -2.5$,
 $T_y(6, 4) \approx \frac{T(6, 2) - T(6, 4)}{-2} = \frac{87 - 80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6, 4)$ to be approximately -3.0°C/m .

- (b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 11.6.9, $D_{\mathbf{u}}T(6, 4) = \nabla T(6, 4) \cdot \mathbf{u} = T_x(6, 4) \frac{1}{\sqrt{2}} + T_y(6, 4) \frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}T(6, 4) \approx (3.5) \frac{1}{\sqrt{2}} + (-3.0) \frac{1}{\sqrt{2}} = \frac{0.5}{\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6, 4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35^\circ\text{C}/\text{m}$.

Alternatively, we can use Definition 11.6.2: $D_{\mathbf{u}}T(6, 4) = \lim_{h \rightarrow 0} \frac{T\left(6 + h \frac{1}{\sqrt{2}}, 4 + h \frac{1}{\sqrt{2}}\right) - T(6, 4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}T(6, 4) \approx \frac{T(8, 6) - T(6, 4)}{2\sqrt{2}} = \frac{80 - 80}{2\sqrt{2}} = 0$,

$$D_{\mathbf{u}}T(6, 4) \approx \frac{T(4, 2) - T(6, 4)}{-2\sqrt{2}} = \frac{74 - 80}{-2\sqrt{2}} = \frac{3}{\sqrt{2}}. \text{ Averaging these values, we have}$$

$$D_{\mathbf{u}}T(6, 4) \approx \frac{3}{2\sqrt{2}} \approx 1.1^\circ\text{C}/\text{m}.$$

- (c) $T_{xy}(x, y) = \frac{\partial}{\partial y} [T_x(x, y)] = \lim_{h \rightarrow 0} \frac{T_x(x, y+h) - T_x(x, y)}{h}$, so $T_{xy}(6, 4) = \lim_{h \rightarrow 0} \frac{T_x(6, 4+h) - T_x(6, 4)}{h}$

which we can estimate with $h = \pm 2$. We have $T_x(6, 4) \approx 3.5$ from part (a), but we will also need values for $T_x(6, 6)$ and $T_x(6, 2)$. If we use $h = \pm 2$ and the values given in the table, we have

$$T_x(6, 6) \approx \frac{T(8, 6) - T(6, 6)}{2} = \frac{80 - 75}{2} = 2.5, \quad T_x(6, 6) \approx \frac{T(4, 6) - T(6, 6)}{-2} = \frac{68 - 75}{-2} = 3.5.$$

Averaging these values, we estimate $T_x(6, 6) \approx 3.0$. Similarly,

$$T_x(6, 2) \approx \frac{T(8, 2) - T(6, 2)}{2} = \frac{90 - 87}{2} = 1.5, \quad T_x(6, 2) \approx \frac{T(4, 2) - T(6, 2)}{-2} = \frac{74 - 87}{-2} = 6.5.$$

Averaging these values, we estimate $T_{xx}(6, 2) \approx 4.0$. Finally, we estimate $T_{xy}(6, 4)$:

$$T_{xy}(6, 4) \approx \frac{T_x(6, 6) - T_x(6, 4)}{2} = \frac{3.0 - 3.5}{2} = -0.25,$$

$$T_{xy}(6, 4) \approx \frac{T_{xx}(6, 2) - T_x(6, 4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25. \text{ Averaging these values, we have}$$

$$T_{xy}(6, 4) \approx -0.25.$$

12. From the table, $T(6, 4) = 80$, and from Exercise 11 we estimated $T_x(6, 4) \approx 3.5$ and $T_y(6, 4) \approx -3.0$. The linear approximation then is

$$\begin{aligned} T(x, y) &\approx T(6, 4) + T_x(6, 4)(x - 6) + T_y(6, 4)(y - 4) \approx 80 + 3.5(x - 6) - 3(y - 4) \\ &= 3.5x - 3y + 71 \end{aligned}$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1^\circ\text{C}$.

$$13. f(x, y) = \sqrt{2x + y^2} \Rightarrow f_x = \frac{1}{2}(2x + y^2)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^2}},$$

$$f_y = \frac{1}{2}(2x + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{2x + y^2}}$$

$$14. u = e^{-r} \sin 2\theta \Rightarrow u_r = -e^{-r} \sin 2\theta, u_\theta = 2e^{-r} \cos 2\theta$$

$$15. g(u, v) = u \tan^{-1} v \Rightarrow g_u = \tan^{-1} v, g_v = \frac{u}{1 + v^2}$$

$$16. w = \frac{x}{y-z} \Rightarrow w_x = \frac{1}{y-z}, w_y = x(-1)(y-z)^{-2} = -\frac{x}{(y-z)^2},$$

$$w_z = x(-1)(y-z)^{-2}(-1) = \frac{x}{(y-z)^2}$$

$$17. T(p, q, r) = p \ln(q + e^r) \Rightarrow T_p = \ln(q + e^r), T_q = \frac{p}{q + e^r}, T_r = \frac{pe^r}{q + e^r}$$

$$18. C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D \Rightarrow$$

$$\partial C / \partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35), \partial C / \partial S = 1.34 - 0.01T, \text{ and } \partial C / \partial D = 0.016. \text{ When}$$

$$T = 10, S = 35, \text{ and } D = 100 \text{ we have } \partial C / \partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35 - 35) \approx 3.587,$$

thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C / \partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C / \partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

$$19. f(x, y) = 4x^3 - xy^2 \Rightarrow f_x = 12x^2 - y^2, f_y = -2xy, f_{xx} = 24x, f_{yy} = -2x, \text{ and } f_{xy} = f_{yx} = -2y.$$

$$20. z = xe^{-2y} \Rightarrow z_x = e^{-2y}, z_y = -2xe^{-2y}, z_{xx} = 0, z_{yy} = 4xe^{-2y}, \text{ and } z_{xy} = z_{yx} = -2e^{-2y}.$$

$$21. f(x, y, z) = x^k y^l z^m \Rightarrow f_x = kx^{k-1} y^l z^m, f_y = lx^k y^{l-1} z^m, f_z = mx^k y^l z^{m-1},$$

$$f_{xx} = k(k-1)x^{k-2} y^l z^m, f_{yy} = l(l-1)x^k y^{l-2} z^m, f_{zz} = m(m-1)x^k y^l z^{m-2}, f_{xy} = f_{yx} = kly^{l-1} x^{k-1} z^m,$$

$$f_{xz} = f_{zx} = kmx^{k-1} y^l z^{m-1}, \text{ and } f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}.$$

$$22. v = r \cos(s + 2t) \Rightarrow v_r = \cos(s + 2t), v_s = -r \sin(s + 2t), v_t = -2r \sin(s + 2t), v_{rr} = 0,$$

$$v_{ss} = -r \cos(s + 2t), v_{tt} = -4r \cos(s + 2t), v_{rs} = v_{sr} = -\sin(s + 2t), v_{rt} = v_{tr} = -2 \sin(s + 2t), \text{ and}$$

$$v_{st} = v_{ts} = -2r \cos(s + 2t).$$

$$23. u = x^y \Rightarrow u_x = yx^{y-1}, u_y = x^y \ln x \text{ and } (x/y)u_x + (\ln x)^{-1}u_y = x^y + x^y = 2u.$$

$$24. \rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \rho_{xx} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}.$$

By symmetry, $\rho_{yy} = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$ and $\rho_{zz} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}}$. Thus

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = 2 \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{(x^2 + y^2 + z^2)^{1/2}} = \frac{2}{\rho}.$$

$$25. (a) z_x = 6x + 2 \Rightarrow z_x(1, -2) = 8 \text{ and } z_y = -2y \Rightarrow z_y(1, -2) = 4, \text{ so an equation of the tangent plane is}$$

$$z - 1 = 8(x - 1) + 4(y + 2) \text{ or } z = 8x + 4y + 1.$$

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1 \rangle$. Then parametric equations for the normal line there are $x = 1 + 8t, y = -2 + 4t, z = 1 - t$, and symmetric equations are

$$\frac{x-1}{8} = \frac{y+2}{4} = \frac{z-1}{-1}.$$

$$26. (a) z_x = e^x \cos y \Rightarrow z_x(0, 0) = 1 \text{ and } z_y = -e^x \sin y \Rightarrow z_y(0, 0) = 0, \text{ so an equation of the tangent plane is}$$

$$z - 1 = 1(x - 0) + 0(y - 0) \text{ or } z = x + 1.$$

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1 \rangle$. Then parametric equations for the normal line there are $x = t, y = 0, z = 1 - t$, and symmetric equations are $x = 1 - z, y = 0$.

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2, -1, 1) = 4$, $F_y(2, -1, 1) = -4$, $F_z(2, -1, 1) = -6$. From Equation 11.6.19, an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or equivalently $2x - 2y - 3z = 3$.

(b) From Equation 11.6.20, symmetric equations for the normal line are $\frac{x-2}{4} = \frac{y+1}{-4} = \frac{z-1}{-6}$.

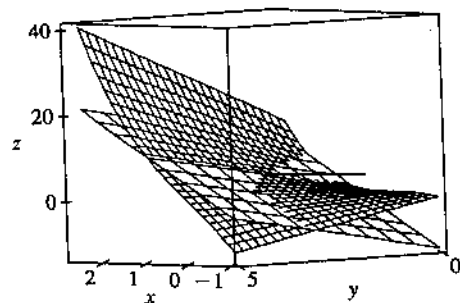
28. (a) Let $F(x, y, z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2$. From Equation 11.6.19, an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or equivalently $x + y + z = 3$.

(b) From Equations 11.6.20, symmetric equations for the normal line are $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$ or equivalently $x = y = z$.

29. (a) $\mathbf{r}(u, v) = (u + v)\mathbf{i} + u^2\mathbf{j} + v^2\mathbf{k}$ and the point $(3, 4, 1)$ corresponds to $u = 2, v = 1$. Then $\mathbf{r}_u = \mathbf{i} + 2u\mathbf{j} \Rightarrow \mathbf{r}_u(2, 1) = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{r}_v = \mathbf{i} + 2v\mathbf{k} \Rightarrow \mathbf{r}_v(2, 1) = \mathbf{i} + 2\mathbf{k}$. A normal vector to the surface at $(3, 4, 1)$ is $\mathbf{r}_u \times \mathbf{r}_v = 8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, so an equation of the tangent plane there is $8(x - 3) - 2(y - 4) - 4(z - 1) = 0$ or equivalently $4x - y - 2z = 6$.

(b) A direction vector for the normal line through $(3, 4, 1)$ is $8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, so a vector equation is $\mathbf{r}(t) = (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + t(8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$, and the corresponding parametric equations are $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$.

30. Let $f(x, y) = x^3 + 2xy$. Then $f_x(x, y) = 3x^2 + 2y$ and $f_y(x, y) = 2x$, so $f_x(1, 2) = 7$, $f_y(1, 2) = 2$ and an equation of the tangent plane is $z - 5 = 7(x - 1) + 2(y - 2)$ or $7x + 2y - z = 6$. The normal line is given by $\frac{x-1}{7} = \frac{y-2}{2} = \frac{z-5}{-1}$ or $x = 7t + 1, y = 2t + 2, z = -t + 5$.



31. $F(x, y, z) = x^2 + y^2 + z^2$, $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle = k\langle 2, 1, -3 \rangle$ or $x_0 = k, y_0 = \frac{1}{2}k$ and $z_0 = -\frac{3}{2}k$. But $x_0^2 + y_0^2 + z_0^2 = 1$, so $\frac{7}{2}k^2 = 1$ and $k = \pm\sqrt{\frac{2}{7}}$. Hence there are two such points: $(\pm\sqrt{\frac{2}{7}}, \pm\frac{1}{\sqrt{14}}, \mp\frac{3}{\sqrt{14}})$.

32. $z = x^2 \tan^{-1} y \Rightarrow dz = (2x \tan^{-1} y) dx + [x^2/(y^2 + 1)] dy$

33. $f(x, y, z) = x^3 \sqrt{y^2 + z^2} \Rightarrow f_x(x, y, z) = 3x^2 \sqrt{y^2 + z^2}$, $f_y(x, y, z) = \frac{yx^3}{\sqrt{y^2 + z^2}}$, and

$$f_z(x, y, z) = \frac{zx^3}{\sqrt{y^2 + z^2}}, \text{ so } f(2, 3, 4) = 8(5) = 40, f_x(2, 3, 4) = 3(4)\sqrt{25} = 60, f_y(2, 3, 4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}, \text{ and}$$

$$f_z(2, 3, 4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}. \text{ Then the linear approximation of } f \text{ at } (2, 3, 4) \text{ is}$$

$$\begin{aligned} f(x, y, z) &\approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) \\ &= 40 + 60(x - 2) + \frac{24}{5}(y - 3) + \frac{32}{5}(z - 4) = 60x + \frac{24}{5}y + \frac{32}{5}z - 120 \end{aligned}$$

Then

$$\begin{aligned} (1.98)^3 \sqrt{(3.01)^2 + (3.97)^2} &= f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 \\ &= 38.656 \end{aligned}$$

34. (a) $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = \frac{1}{2}y dx + \frac{1}{2}x dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017 \text{ m}^2$ or 170 cm^2 .

- (b) $z = \sqrt{x^2 + y^2}$, $dz = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026 \text{ m}$ or 0.26 cm .

35. $\frac{dw}{dt} = \frac{1}{2\sqrt{x}} (2e^{2t}) + \frac{2y}{z} (3t^2 + 4) + \frac{-y^2}{z^2} (2t) = e^t + \frac{2y}{z} (3t^2 + 4) - 2t \frac{y^2}{z^2}$

36. $\frac{\partial z}{\partial u} = (-y \sin xy - y \sin x)(2u) + (-x \sin xy + \cos x) = \cos x - 2uy \sin x - (\sin xy)(x + 2uy)$,

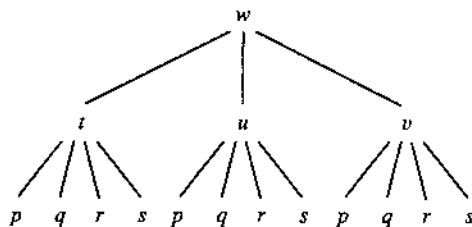
$\frac{\partial z}{\partial v} = (-y \sin xy - y \sin x)(1) + (-x \sin xy + \cos x)(-2v) = -2v \cos x + (\sin xy)(2vx - y) - y \sin x$

37. By the Chain Rule, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1, 2) = 3$ and $y = h(1, 2) = 6$, so

$\frac{\partial z}{\partial s} = f_x(3, 6)g_s(1, 2) + f_y(3, 6)h_s(1, 2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$, so

$\frac{\partial z}{\partial t} = f_x(3, 6)g_t(1, 2) + f_y(3, 6)h_t(1, 2) = (7)(4) + (8)(10) = 108$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial p} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p}$$

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial q} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial r} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial s} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial s}$$

39. $\frac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$, $\frac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2)$ [where $f' = \frac{df}{d(x^2 - y^2)}$]. Then

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x.$$

40. $A = \frac{1}{2}xy \sin \theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$\frac{dA}{dt} = \frac{1}{2} \left[(y \sin \theta) \frac{dx}{dt} + (x \sin \theta) \frac{dy}{dt} + (xy \cos \theta) \frac{d\theta}{dt} \right]$. So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$,

$\frac{dA}{dt} = \frac{1}{2} [(25)(3) + (20)(-2) + (1000\sqrt{3})(0.05)] = \frac{35 + 50\sqrt{3}}{2} \approx 60.8 \text{ in}^2/\text{s}$.

41. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} y + \frac{\partial z}{\partial v} \frac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{2y}{x^3} \frac{\partial z}{\partial v} + \frac{-y}{x^2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\ &= \frac{2y}{x^3} \frac{\partial z}{\partial v} + y \left(\frac{\partial^2 z}{\partial u^2} y + \frac{\partial^2 z}{\partial v \partial u} \frac{-y}{x^2} \right) + \frac{-y}{x^2} \left(\frac{\partial^2 z}{\partial v^2} \frac{-y}{x^2} + \frac{\partial^2 z}{\partial u \partial v} y \right) \\ &= \frac{2y}{x^3} \frac{\partial z}{\partial v} + y^2 \frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2} \frac{\partial^2 z}{\partial u \partial v} + \frac{y^2}{x^4} \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\frac{\partial z}{\partial y} = x \frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial z}{\partial v}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= x \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + \frac{1}{x} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right) = x \left(\frac{\partial^2 z}{\partial u^2} x + \frac{\partial^2 z}{\partial v \partial u} \frac{1}{x} \right) + \frac{1}{x} \left(\frac{\partial^2 z}{\partial v^2} \frac{1}{x} + \frac{\partial^2 z}{\partial u \partial v} x \right) \\ &= x^2 \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{x^2} \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Thus

$$\begin{aligned}x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x} \frac{\partial z}{\partial v} + x^2 y^2 \frac{\partial^2 z}{\partial u^2} - 2y^2 \frac{\partial^2 z}{\partial u \partial v} + \frac{y^2}{x^2} \frac{\partial^2 z}{\partial v^2} - x^2 y^2 \frac{\partial^2 z}{\partial u^2} - 2y^2 \frac{\partial^2 z}{\partial u \partial v} - \frac{y^2}{x^2} \frac{\partial^2 z}{\partial v^2} \\ &= \frac{2y}{x} \frac{\partial z}{\partial v} - 4y^2 \frac{\partial^2 z}{\partial u \partial v} = 2v \frac{\partial z}{\partial v} - 4uv \frac{\partial^2 z}{\partial u \partial v}\end{aligned}$$

since $y = xv = \frac{uv}{y}$ or $y^2 = uv$.

42. $F(x, y, z) = e^{xyz} - yz^4 - x^2 z^3 = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz} - 2xz^3}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \frac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}$

and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz} - z^4}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \frac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}$.

43. $\nabla f = \left\langle z^2 \sqrt{y} e^{x\sqrt{y}}, \frac{xz^2 e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}} \right\rangle = ze^{x\sqrt{y}} \left\langle z\sqrt{y}, \frac{xz}{2\sqrt{y}}, 2 \right\rangle$

44. (a) By Theorem 11.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}} f = |\nabla f| \cos \theta$ (see the proof of Theorem 11.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}} f = |\nabla f| \cos \theta = \frac{1}{2} |\nabla f| \Leftrightarrow \cos \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$.

45. $\nabla f = \langle 1/\sqrt{x}, -2y \rangle$, $\nabla f(1, 5) = \langle 1, -10 \rangle$, $\mathbf{u} = \frac{1}{5} \langle 3, -4 \rangle$. Then $D_{\mathbf{u}} f(1, 5) = \frac{43}{5}$.

46. $\nabla f = \langle 2xy + \sqrt{1+z}, x^2, x/(2\sqrt{1+z}) \rangle$, $\nabla f(1, 2, 3) = \langle 6, 1, \frac{1}{4} \rangle$, and $\mathbf{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$. Then $D_{\mathbf{u}} f(1, 2, 3) = \frac{25}{6}$.

47. $\nabla f = \langle 2xy, x^2 + 1/(2\sqrt{y}) \rangle$, $|\nabla f(2, 1)| = |\langle 4, \frac{9}{2} \rangle|$. Thus the maximum rate of change of f at $(2, 1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\langle 4, \frac{9}{2} \rangle$.