Solutions: Probability Homework 1
Biochem 21a, Fall 2004

1) a) $P(A) = \frac{3}{13}$

b) $P(A \cap B) = P(\text{drawing a red jack}) = \frac{1}{26}$

c) $P(A \cup B) = P(\text{drawing J, Q, or K or red 9, or red 10})$

$$= \frac{3}{13} + \frac{4}{52} = \frac{4}{13}$$

d) $P(C \cup D) = 1$

e) $P(C \cap D) = 0$

2) a)

HHHH	HTHH	THHH	TTHH
HHHT	HTHT	THHT	TTHT
HHTH	HTTH	THTH	TTTH
HHTT	HTTT	THTT	TTTT

b) i) $P(A) = \frac{5}{16}$

ii) $P(A \cap B) = 0$

iii) $P(B) = \frac{11}{16}$

iv) $P(A \cup C) = P(\text{at least three heads or heads on } 3^{rd} \text{ toss})$

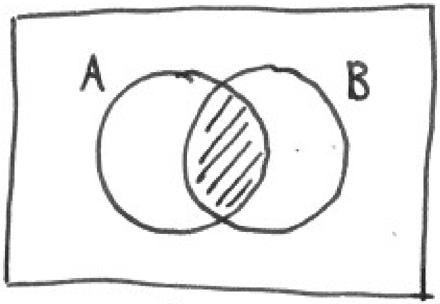
$$= \frac{9}{16}$$

v) $P(D) = \binom{4}{3}\frac{1}{16} = \frac{1}{4}$

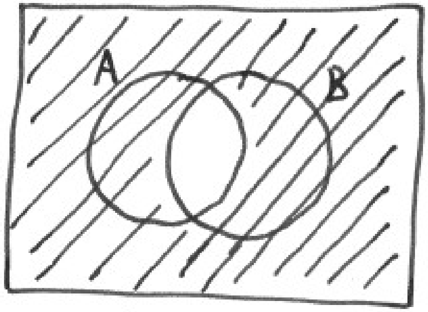
vi) $P(A \cap C) = \frac{1}{4}$

vii) $P(B \cup D) = P(B) = \frac{11}{16}$

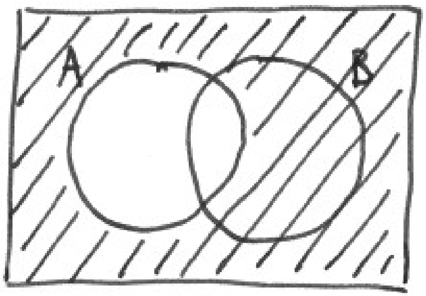
3) a)



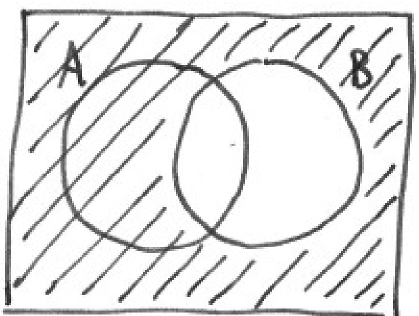
$A \cap B$



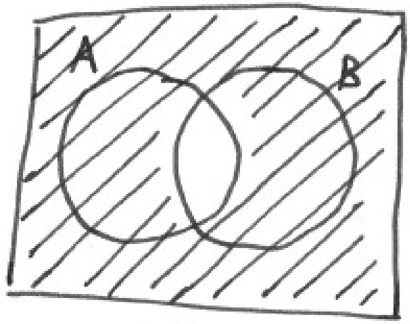
$(A \cap B)^c$



A^c



B^c

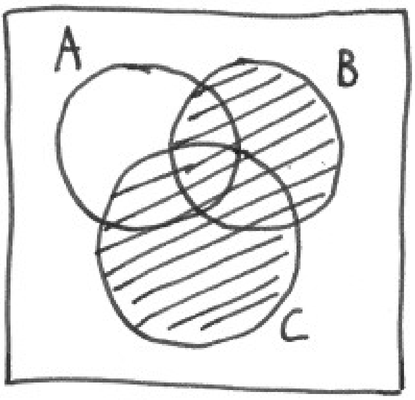


$A^c \cup B^c$

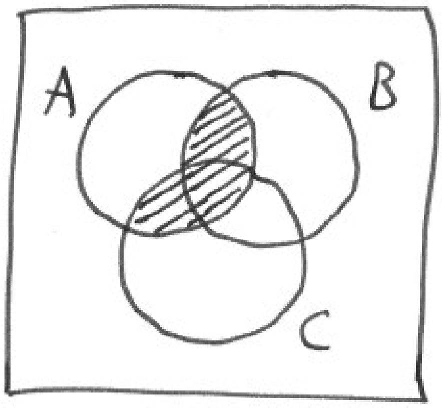
Because the two sets depicted in the diagram on the right are equal, we have

$$(A \cap B)^c = A^c \cup B^c$$

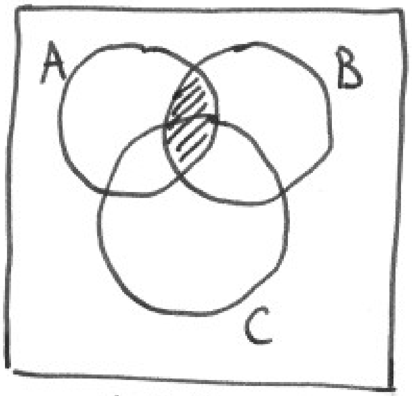
3) b)



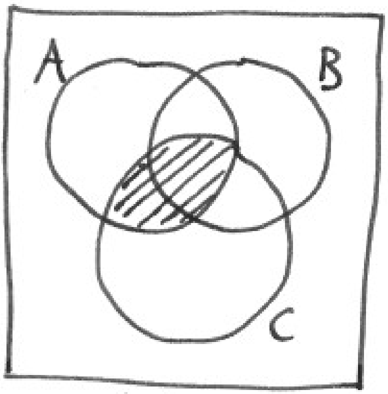
$B \cup C$



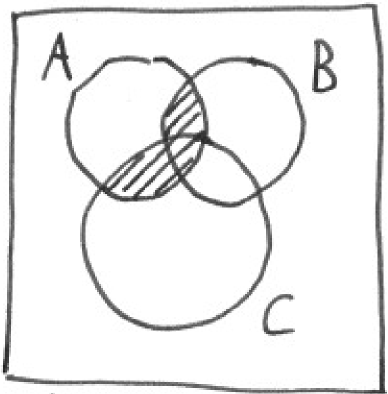
$A \cap (B \cup C)$



$A \cap B$



$A \cap C$

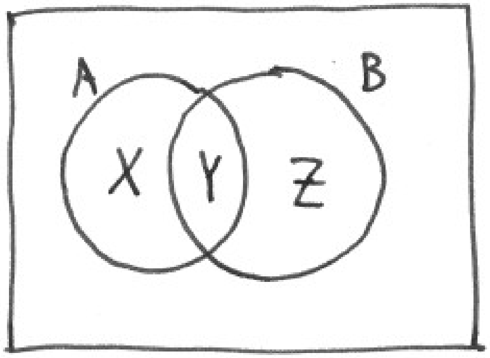


$(A \cap B) \cup (A \cap C)$

Because the two sets depicted on the right are equal, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4) Let X, Y, and Z be subsets of Ω as shown:



Then $P(A) = P(X) + P(Y)$
and $P(B) = P(Y) + P(Z)$.

Further,

$$\begin{aligned} P(A \cup B) &= P(X) + P(Y) + P(Z) \\ &= P(A) + P(B) - P(Y) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Part 1c)

$$\left. \begin{aligned} P(A \cup B) &= \tfrac{4}{13} \\ P(A) &= \tfrac{3}{13} \\ P(B) &= \tfrac{3}{26} \\ P(A \cap B) &= \tfrac{1}{26} \end{aligned} \right\} \quad \frac{3}{13} + \frac{3}{26} - \frac{1}{26} = \frac{4}{13} \ \checkmark$$

Part 1d)

$$\left. \begin{aligned} P(C \cup D) &= 1 \\ P(C) &= \tfrac{1}{4} \\ P(D) &= \tfrac{3}{4} \\ P(C \cap D) &= 0 \end{aligned} \right\} \quad \frac{1}{4} + \frac{3}{4} - 0 = 1 \ \checkmark$$

5) Explanation 1 of Monty Hall:

The contestant chooses a door, which has $\frac{1}{3}$ chance of being in front of afabulous prize. The chance that the prize is behind the other two doors is $\frac{2}{3}$. When Monty Hall opens a door, the contestant's door <u>still has</u> a $\frac{1}{3}$ chance of being the winning door. Consequently the remaining door (not chosen by the contestant or opened by Monty Hall) has a $\frac{2}{3}$ chance of being the winning door.

Explanation 2 of Monty Hall:

Label the doors A, B, and C, and assume, without loss of generality, that the contestant chooses door A. Then

P(A is the winning door and Monty opens B)
$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

P(A is the winning door and Monty opens C)
$= \frac{1}{6}$

P(B is winning door and Monty opens C) $= \frac{1}{3}$

P(C is winning door and Monty opens B) $= \frac{1}{3}$

Only if the first or second event occurs should the contestant stick with her original choice.

5) a) You chose door 1. Before Monty acts, we have

$P(\text{prize behind } 1) = \frac{1}{3}$

$P(\text{prize behind } 2) = \frac{1}{3}$

$P(\text{prize behind } 3) = \frac{1}{3}$

Thus we have the following table of probabilities that the prize is behind a given door and Monty makes a given action:

Prize behind: / Monty opens:	Door 1	Door 2	Door 3	No Door
Door 1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
Door 2	0	0	$\frac{1}{9}$	$\frac{2}{9}$
Door 3	0	$\frac{1}{9}$	0	$\frac{2}{9}$

Thus $P(\text{prize behind } 1 \mid \text{Monty opens } 3)$

$$= \frac{P(\text{prize behind } 1 \text{ and Monty opens } 3)}{P(\text{Monty opens } 3)}$$

$$= \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{9}} = \frac{3}{7}$$

5) a) (continued)

Similarly,

$$P(\text{prize behind } 2 \mid \text{Monty opens } 3)$$

$$= \frac{P(\text{prize behind } 2 \text{ and Monty opens } 3)}{P(\text{Monty opens } 3)}$$

$$= \frac{\frac{1}{9}}{\frac{1}{12} + \frac{1}{9}} = \frac{4}{7}$$

Therefore you should switch to door 2.

5) b) As in (a), we have the following table:

Monty opens:

Prize behind:	Door 1	Door 2	Door 3	No Door
Door 1	0	$\frac{1}{6} p_1$	$\frac{1}{6} p_1$	$\frac{1}{3}(1-p_1)$
Door 2	0	0	$\frac{1}{3} p_2$	$\frac{1}{3}(1-p_2)$
Door 3	0	$\frac{1}{3} p_2$	0	$\frac{1}{3}(1-p_2)$

Then

$$P(\text{prize behind } 1 \mid \text{Monty opens } 3) = \frac{\frac{1}{6} p_1}{\frac{1}{6} p_1 + \frac{1}{3} p_2} = \frac{p_1}{p_1 + 2p_2}$$

and $P(\text{prize behind } 2 \mid \text{Monty opens } 3) = \frac{2p_2}{p_1 + p_2}$

You should switch to door 2 if

$$\frac{2p_2}{p_1 + 2p_2} > \frac{p_1}{p_1 + 2p_2}$$

$$\Longrightarrow 2p_2 > p_1$$

6) Note that

$$P(BBYY) = \frac{4}{8}\cdot\frac{3}{7}\cdot\frac{4}{6}\cdot\frac{3}{5} = \frac{3}{35}$$

and

$$P(BYBY) = \frac{4}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5} = \frac{3}{35}.$$

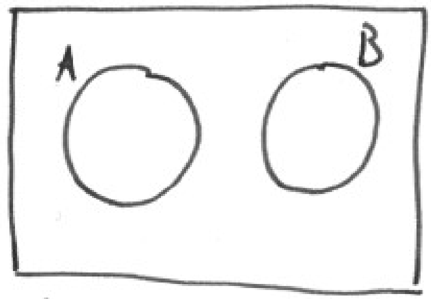
In particular, note that the probability of drawing 2 blue balls and 2 yellow balls ~~is always $\frac{3}{35}$ because the numerator is always $4\cdot4\cdot3\cdot3$ and the denom~~ in a particular order is always $\frac{3}{35}$ because the numerator is always $4\cdot4\cdot3\cdot3$ and the denominator is always $8\cdot7\cdot6\cdot5$.

IF A is the event of drawing 2 blue balls and 2 yellow balls, then $P(A) = \binom{4}{2}\cdot\frac{3}{35} = \frac{18}{35}$.

Then the probability of failing the first 14 times and succeeding on the 15^{th} attempt is

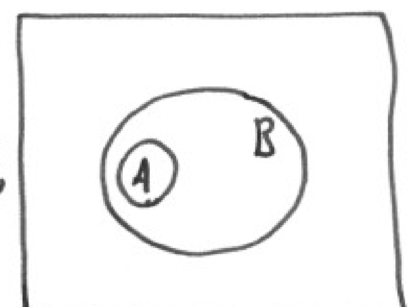
$$(1-P(A))^{14}(P(A)) = \left(\frac{17}{35}\right)^{14}\left(\frac{18}{35}\right)$$

7) a) If A and B are mutually exclusive, then A and B cannot be independent unless $P(A)=0$ or $P(B)=0$.



In particular, $A \cap B = \emptyset$, so $P(A \cap B) = 0$. If A and B are independent, then $P(A \cap B) = P(A)\,P(B)$. In other words, A and B can be mutually exclusive and independent only when $A = \emptyset$ or $B = \emptyset$.

b) If $A \subset B$ then $A \cap B = A$.



If A and B are independent, then $P(A \cap B) = P(A)\,P(B)$

$$\Rightarrow P(A) = P(A)\,P(B).$$

Thus we must have either:

$* \; P(A) = 0 \Rightarrow A = \emptyset$

or $* \; P(B) = 1 \Rightarrow B = \Omega$

8) a) $P(\text{failure} \mid A) = 1/10$ $\qquad P(A) = 3/5$

$P(\text{failure} \mid B) = 1/20$ $\qquad P(B) = 2/5$

Denote the 1^{st}, 2^{nd}, and 3^{rd} razors tested by 1, 2, and 3 respectively.

$$\begin{aligned} P(1 \text{ and } 2 \text{ fail}) &= P((1 \text{ and } 2 \text{ fail}) \text{ and } A) \\ &\quad + P((1 \text{ and } 2 \text{ fail}) \text{ and } B) \\ &= \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{3}{5} + \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{2}{5} \\ &= \frac{7}{1000} \end{aligned}$$

$$\begin{aligned} &P(3 \text{ fails} \mid 1 \text{ and } 2 \text{ fail}) \\ &= \frac{P(3 \text{ fails and } (1 \text{ and } 2 \text{ fail}))}{P(1 \text{ and } 2 \text{ fail})} \\ &= \frac{P(1,2,3 \text{ fail and } A) + P(1,2,3 \text{ fail and } B)}{P(1 \text{ and } 2 \text{ fail})} \\ &= \frac{\left(\frac{1}{10}\right)^3 \cdot \left(\frac{3}{5}\right) + \left(\frac{1}{20}\right)^3 \cdot \left(\frac{2}{5}\right)}{\frac{7}{1000}} = \frac{13}{140} \end{aligned}$$

8) b) Failure of razor 1 is not independent of failure of razor 2: If razor 1 fails, it is more likely the bag is from plant A, which means that razor 2 is more likely to fail.