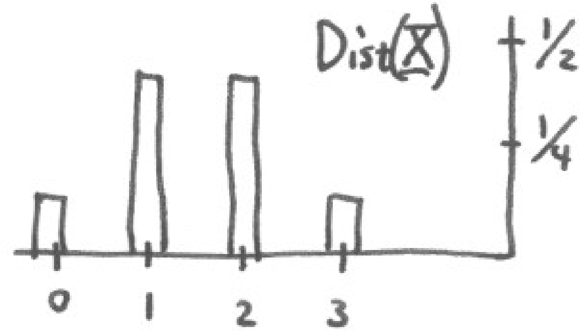
Probability PS 2 Solutions
Biochem 21a Fall 2004

1) $\Omega = \left\{ \begin{array}{ll} HHH & THH \\ HHT & THT \\ HTH & TTH \\ HTT & TTT \end{array} \right\}$

a) $\text{Range}(X) = \{0, 1, 2, 3\}$

$P(X = 0) = 1/8 = P(X = 3)$

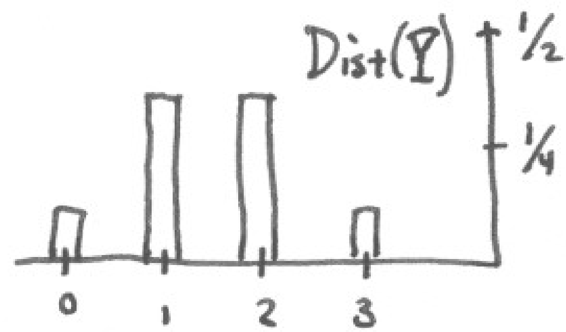
$P(X = 1) = 3/8 = P(X = 2)$



$\text{Range}(Y) = \{0, 1, 2, 3\}$

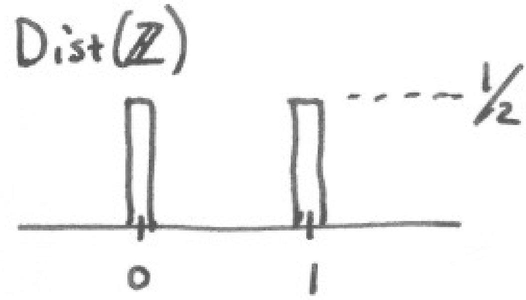
$P(Y = 0) = 1/8 = P(Y = 3)$

$P(Y = 1) = 3/8 = P(Y = 2)$



$\text{Range}(Z) = \{0, 1\}$

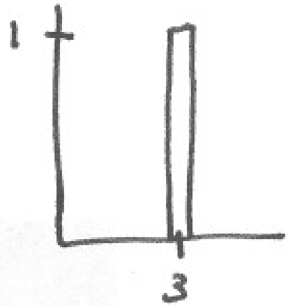
$P(Z = 0) = P(Z = 1) = 1/2$



X and Y have the same distributions, but they are not equal because $X(HHH) = 3$ and $Y(HHH) = 0$, and equal random variables must have equal values at the same outcome ω in Ω.

1) b) Let $X = 0$.

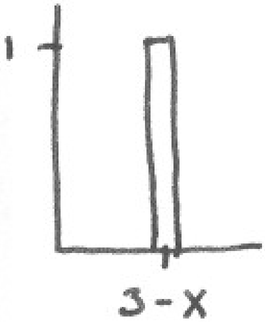
Then $Y = 3$.



$\mathrm{Dist}(Y \mid X = 0)$

Let $X = x$.

Then $Y = 3 - x$



$\mathrm{Dist}(Y \mid X = x)$

No, X and Y are not independent.

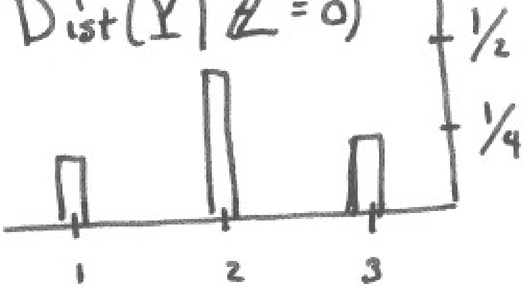
c) Let $Z = 0$

Then $P(Y = 1) = 1/4$

$P(Y = 2) = 1/2$

$P(Y = 3) = 1/4$

$\mathrm{Dist}(Y \mid Z = 0)$

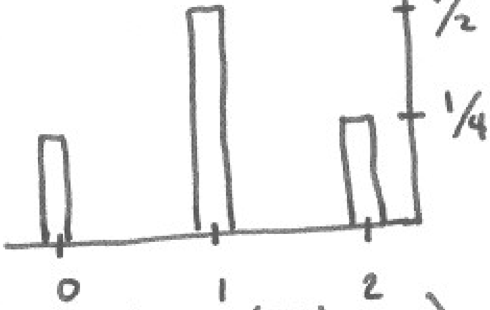


Let $Z = 1$

Then $P(Y = 0 \mid Z = 1) = 1/4$

$P(Y = 1 \mid Z = 1) = 1/2$

$P(Y = 2 \mid Z = 1) = 1/4$



$\mathrm{Dist}(Y \mid Z = 1)$

1) d) $\text{Range}[(X, Y)] = \{(0,3), (1,2), (2,1), (3,0)\}$

$$P(X=0, Y=3) = P(X=3, Y=0) = 1/8$$

$$P(X=1, Y=2) = P(X=2, Y=1) = 3/8$$

e) $\text{Range}[(X, Y, Z)] = \{(0,3,0), (1,2,0), (1,2,1), (2,1,0), (2,1,1), (3,0,1)\}$

$$P(X=0, Y=3, Z=0) = P(X=3, Y=0, Z=1) = 1/8$$

$$P(X=1, Y=2, Z=0) = 1/4$$

$$P(X=1, Y=2, Z=1) = 1/8$$

$$P(X=2, Y=1, Z=0) = 1/8$$

$$P(X=2, Y=1, Z=1) = 1/4$$

2) a) $\text{Range}(X) = \{0, 1, 2, 3\}$

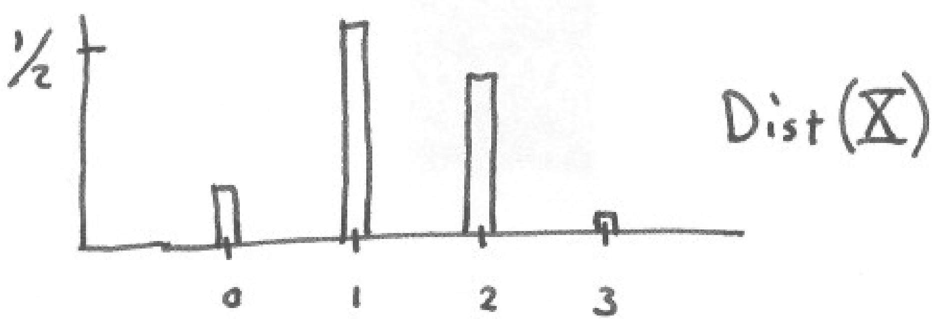
$$P(X=0) = \frac{4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{4}{35}$$

$$P(X=1) = \binom{3}{1} \frac{4 \cdot 3 \cdot 3}{7 \cdot 6 \cdot 5} = \frac{18}{35}$$

$$P(X=2) = \binom{3}{1} \frac{4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{12}{35}$$

$$P(X=3) = \frac{3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5} = \frac{1}{35}$$

(Note that $\frac{4}{35} + \frac{18}{35} + \frac{12}{35} + \frac{1}{35} = 1$, as it should.)



b) $P(1 \le X < 3) = P(X=1) + P(X=2) = \frac{6}{7}$

$P(1 < X \le 3) = P(X=2) + P(X=3) = \frac{13}{35}$

$P(1 < X < 3) = P(X=2) = \frac{12}{35}$

$$P(1 \le X < 3 \mid X < 2) = \frac{P(X=1)}{P(X=0) + P(X=1)} = \frac{18}{4+18} = \frac{9}{11}$$

2) c) $E(X) = 0 \cdot \frac{4}{35} + 1 \cdot \frac{18}{35} + 2 \cdot \frac{12}{35} + 3 \cdot \frac{1}{35}$

$= \frac{9}{7}$

$\text{Range}\left(X - \frac{9}{7}\right) = \left\{ \frac{-9}{7}, \frac{-2}{7}, \frac{5}{7}, \frac{12}{7} \right\}$

$\text{Range}\left[\left(X - \frac{9}{7}\right)^2\right] = \left\{ \frac{81}{49}, \frac{4}{49}, \frac{25}{49}, \frac{144}{49} \right\}$

$P\left[\left(X - \frac{9}{7}\right)^2 = \frac{81}{49}\right] = \frac{4}{35}$

$P\left[\left(X - \frac{9}{7}\right)^2 = \frac{4}{49}\right] = \frac{18}{35}$

$P\left[\left(X - \frac{9}{7}\right)^2 = \frac{25}{49}\right] = \frac{12}{35}$

$P\left[\left(X - \frac{9}{7}\right)^2 = \frac{144}{49}\right] = \frac{1}{35}$

$Var(X) = E\left[\left(X - E(X)\right)^2\right]$

$= \frac{81}{49} \cdot \frac{4}{35} + \frac{4}{49} \cdot \frac{18}{35} + \frac{25}{49} \cdot \frac{12}{35} + \frac{144}{49} \cdot \frac{1}{35}$

$= \frac{24}{49}$

3) a) $\text{Range}(X) = \{1, 2, 3, 4, \ldots\}$

$$P(X \geq 1) = 1$$

$$P(X \geq 2) = P(X \neq 1) = 1 - P(X = 1) = 3/4$$

$$P(X \geq 3) = \frac{3}{4} \cdot \frac{3}{4}$$

$$P(X \geq x) = \left(\frac{3}{4}\right)^{x-1}$$

b) $P(X = 1) = P(X \geq 1) - P(X \geq 2) = 1/4$

$$P(X = 2) = P(X \geq 2) - P(X \geq 3) = 3/4 - 9/16 = 3/16$$

$$P(X = 3) = P(X \geq 3) - P(X \geq 4) = 9/16 - 27/64 = 9/64$$

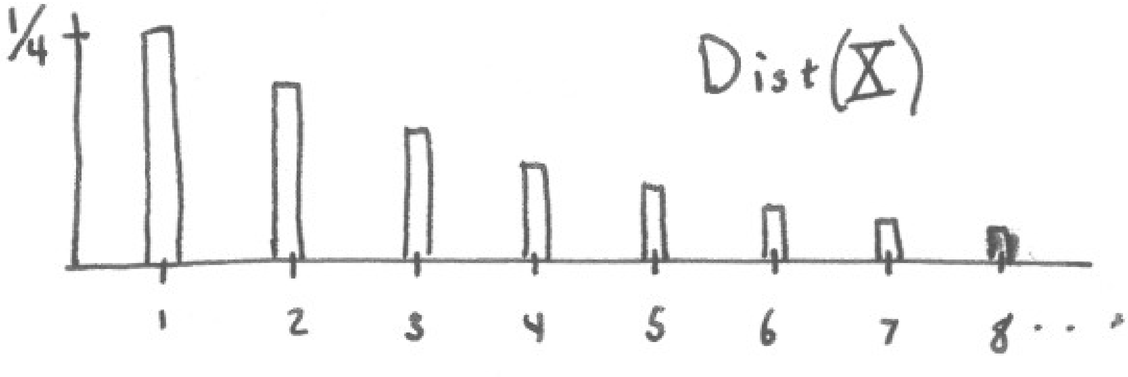
$$P(X = x) = \left(\frac{3}{4}\right)^{x-1} - \left(\frac{3}{4}\right)^{x} = \frac{1}{4}\left(\frac{3}{4}\right)^{x-1}$$

Note that

$$\sum_{i=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{i-1} = \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^{i} = \frac{1}{4} \frac{1}{1 - 3/4} = 1$$

so

$$P(X = 1) + P(X = 2) + P(X = 3) + \ldots = 1$$



3) b) continued

$$E(X) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{4}\left(\frac{3}{4}\right) + 3\cdot\frac{1}{4}\left(\frac{3}{4}\right)^2 + 4\cdot\frac{1}{4}\left(\frac{3}{4}\right)^3 + \cdots$$

$$= \frac{1}{4}\left[1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + 5\left(\frac{3}{4}\right)^4 + \cdots\right]$$

$$= \frac{1}{4}\left[\begin{array}{l} 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \\ \quad + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \\ \qquad + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \\ \qquad\qquad + \cdots \end{array}\right.$$

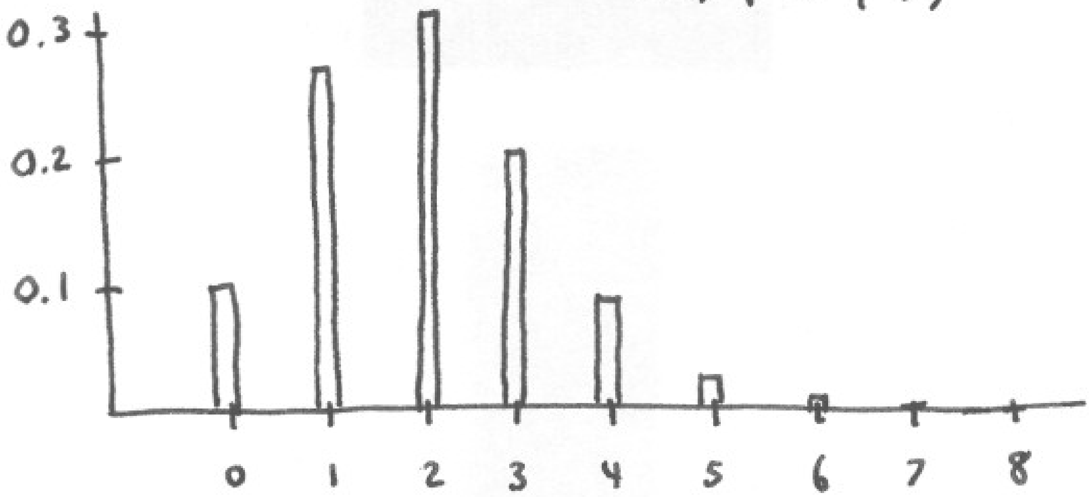
$$= \frac{1}{4}\left[\frac{1}{1-3/4} + \frac{3}{4}\,\frac{1}{1-3/4} + \left(\frac{3}{4}\right)^2\frac{1}{1-3/4} + \cdots\right]$$

$$= 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots$$

$$= \frac{1}{1-3/4} = 4$$

3) c) $\text{Range}(Y_i) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$$P(Y_i = k \mid X \geq i) = \binom{8}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{8-k}$$



$$E(Y_i \mid X \geq i) = 8 \cdot \frac{1}{4} = 2$$

d) $P(Y_i = 0) = \cancel{P(Y_i = 0 \mid X \geq i) + P(X < i)}$

$$\cancel{= 8\left(\frac{3}{4}\right)^8}$$

$$= P(Y_i = 0 \mid X \geq i)P(X \geq i) + P(X < i)$$

$$= \left(\frac{3}{4}\right)^8\left(\frac{3}{4}\right)^{i-1} + 1 - \left(\frac{3}{4}\right)^{i-1}$$

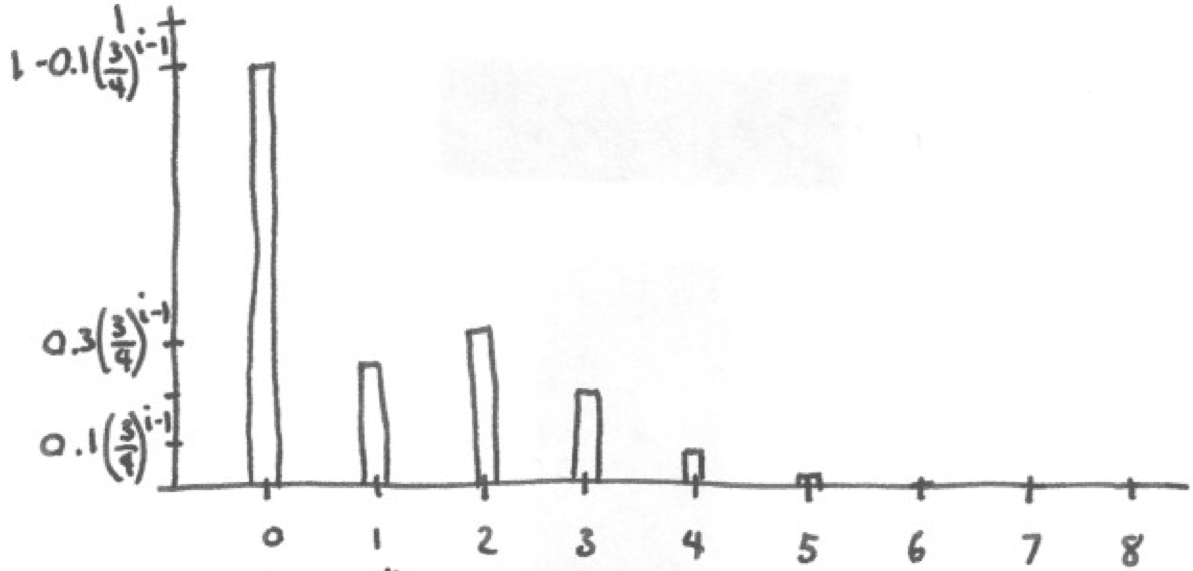
$$= 1 - \left(\frac{3}{4}\right)^{i-1}\left(1 - \left(\frac{3}{4}\right)^8\right) \approx 1 - 0.1\left(\frac{3}{4}\right)^{i-1}$$

Let $k > 0$ and in $\text{Range}(Y_i)$

$$P(Y_i = k) = P(Y_i = k \mid X \geq i)P(X \geq i)$$

$$= \binom{8}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{8-k}\left(\frac{3}{4}\right)^{i-1}$$

3) d) continued



$$E(Y_i) = E(Y_i \mid X \geq i)\,P(X \geq i)$$

$$= 2\left(\frac{3}{4}\right)^{i-1}$$

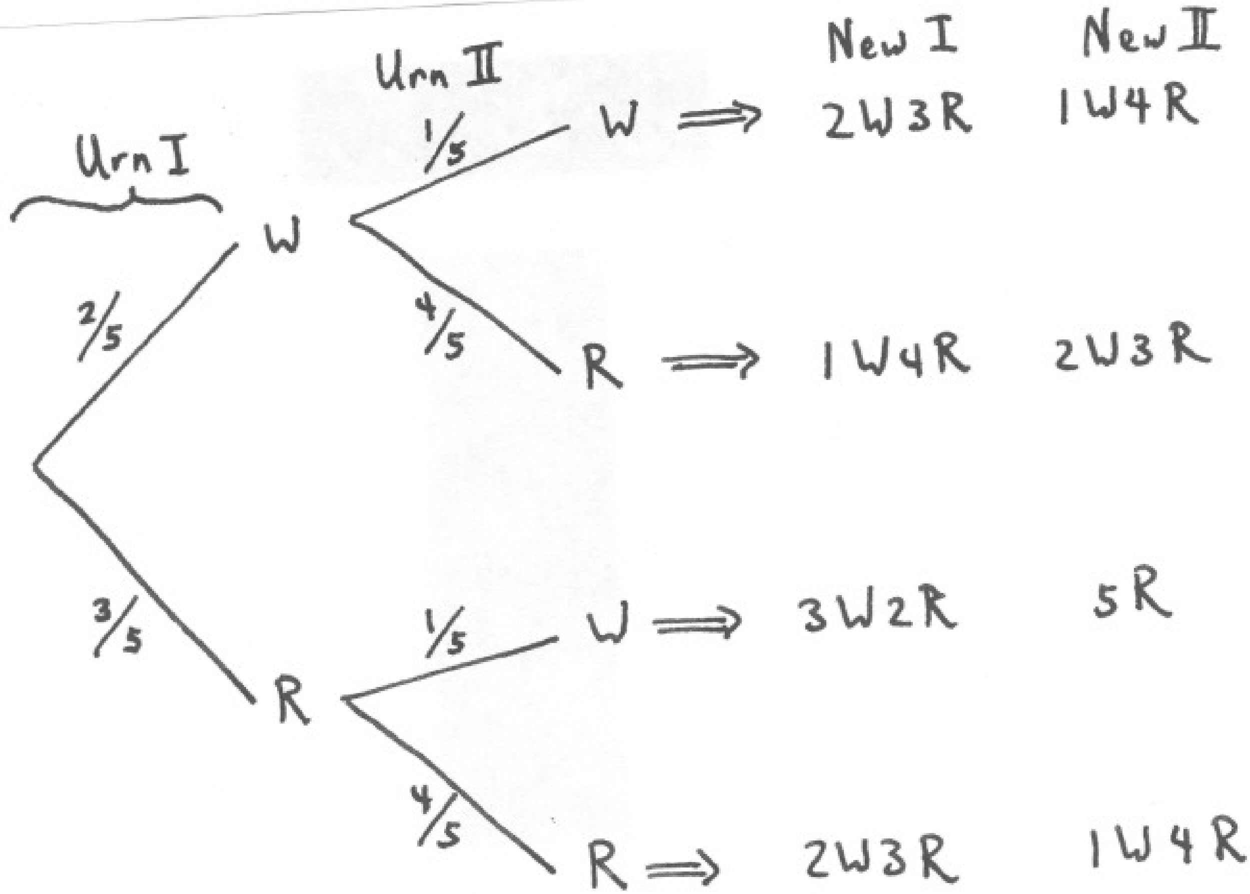
e) $E(Y) = E(Y_1) + E(Y_2) + E(Y_3) + \cdots$

$$= 2 + 2\left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^3 + \cdots$$

$$= 2\,\frac{1}{1-3/4} = 8$$

Yes, $E(Y) = E(X)E(Y_1)$. This makes sense because Marta expects, on average, to infect 2 people each day that she is infectious and not quarantined.

4)



There are two possible outcomes of the switching process:

outcome 1: an urn containing 2W3R and one containing 1W4R

outcome 2: an urn containing 3W2R and one containing 5R

$$P(\text{outcome } 1) = \frac{2}{5} \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{4}{5} = \frac{22}{25}$$

$$P(\text{outcome } 2) = \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25}$$

4) continued

$$\text{Range}(X) = \{0, 1, 2\}$$

$$P(X=0) = P(\text{outcome } 1)\,P(RR \mid \text{outcome } 1) + P(\text{outcome } 2)\,P(RR \mid \text{outcome } 2)$$

$$= \frac{22}{25} \cdot \frac{3}{5} \cdot \frac{4}{5} + \frac{3}{25} \cdot \frac{2}{5} = \frac{294}{625}$$

$$P(X=1) = P(\text{outcome } 1)\,P(WR \text{ or } RW \mid \text{outcome } 1) + P(\text{outcome } 2)\,P(WR \text{ or } RW \mid \text{outcome } 2)$$

$$= \frac{22}{25}\left(\frac{2}{5} \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{1}{5}\right) + \frac{3}{25} \cdot \frac{3}{5} = \frac{287}{625}$$

$$P(X=2) = P(\text{outcome } 1)\,P(WW \mid \text{outcome } 1)$$

$$= \frac{22}{25} \cdot \frac{2}{5} \cdot \frac{1}{5} = \frac{44}{625}$$

$$E(X) = 0 \cdot \frac{294}{625} + 1 \cdot \frac{287}{625} + 2 \cdot \frac{44}{625} = \frac{3}{5}$$

$$\text{Var}(X) = \frac{9}{25} \cdot \frac{294}{625} + \frac{4}{25} \cdot \frac{287}{625} + \frac{49}{25} \cdot \frac{44}{625}$$

$$= \frac{238}{625}$$