Probability Homework 1
Biochem 21a Fall 2004

1. Consider the experiment of drawing a card at random from a standard deck of 52 cards. Then the outcome space Ω is the set of 52 cards. Assume that each card is equally likely to be drawn. Consider the following subsets of Ω:

- $A = \{x|x \text{ is a jack, queen, or king}\}$
- $B = \{x|x \text{ is a 9, 10, or jack, and x is red}\}$
- $C = \{x|x \text{ is a club}\}$
- $D = \{x|x \text{ is a diamond, heart, or spade}\}$

Compute the following:

- (a) $P(A)$
- (b) $P(A \cap B)$
- (c) $P(A \cup B)$
- (d) $P(C \cup D)$
- (e) $P(C \cap D)$

2. A fair coin is tossed 4 times, and a sequence of heads and tails is observed.
(a) List each of the 16 possible sequences that compose the outcome space Ω
(b) Let events A, B, C, and D be given by

- A = at least three heads
- B = at most two heads
- C = heads on the third toss
- D = 1 head and 3 tails

If the probability of each outcome in the outcome space is equally likely, find

- (i) $P(A)$
- (ii) $P(A \cap B)$
- (iii) $P(B)$
- (iv) $P(A \cup C)$
- (v) $P(D)$
- (vi) $P(A \cap C)$
- (vii) $P(B \cup D)$

3. With the aid of Venn Diagrams, show the following:
(a) $(A \cap B)^c = A^c \cup B^c$
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ using Venn Diagrams. Check that this holds for parts (c) and (d) of problem 1.

5. The Monty Hall Problem is famous in probability theory: Monty Hall was a game show host on "Let's make a deal," and he played a game as follows: A contestant was presented with three doors. Behind one door was a fabulous prize, and behind the other two doors were goats. The contestant chose the door behind which she expected to find the fabulous prize. Then Monty Hall (who knew where the prize was) would open one of the two doors she hadn't chosen to reveal a goat, and give the contestant the option of switching her choice to the closed door she hadn't originally chosen. It turns out that contestants had a higher chance of winning

the fabulous prize if they switched than if they stuck with their original choices. (If this seems absurd to you, you can play the game and read an explanation at http://math.ucsd.edu/ crypto/Monty/monty.html) Solve the following variants on the Monty Hall problem:
(a) Suppose instead that Monty Hall knows where the prize is and will not open that door but also has the option of not opening any door. Suppose the probability the host opens a door is 1/2 if the prize is behind the door you pick and the probability the host opens a door is 1/3 if the prize is not behind the door you pick. If you pick door 1 and the host opens door 3, what is the probability the prize is behind door 1? Should you switch to door 2 if you're given that option?
(b) If the probabilities of 1/2 and 1/3 of the host opening a door are replaced by p_1 and p_2, answer the first question in (a). Under what conditions on p_1 and p_2 should you switch to door 2?

6. A bag contains 4 blue and 4 yellow balls. Four balls are chosen at random and without replacement. If 2 of them are blue and 2 are yellow, we stop. If not, the balls are replaced in the bag and 4 balls are again chosen randomly. The process continues until exactly 2 of the 4 balls chosen are blue. What is the probability we have to do this exactly 15 times?

7. (a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain. Note that a diagram and one sentence will suffice.
(b) If $A \subset B$, can A and B ever be independent events? Explain. (Again, a diagram and one sentence will suffice.)

8. A company has two plants, A and B, which make switches for electric razors. Of the switches produced by A, 10% are defective, while 5% of the switches made by B are defective. 60% of the switches are made by A. If a switch is defective, the razor fails to work. The switches are packaged in bags of three and shipped to an assembly plant. If an assembler opens a bag of three switches and puts them into 3 different razors,
(a) what is the probability the 3rd razor fails to work given the 1st and 2nd ones fail?
(b) If E_1 denotes the event that razor 1 fails and E_2 denotes the event that razor 2 fails, are these events independent? Why or why not?