Review Exercises

Chapter 3. Random Variables

1. A fair die is rolled ten times. Write down numerical expressions for:

a) the probability of at least one six in the ten rolls;

b) the expected number of sixes in the ten rolls;

c) the expected sum of the numbers in the ten rolls;

d) the probability of 2 sixes in the first five rolls given 4 sixes in the ten rolls;

e) the probability of getting strictly more sixes in the second five rolls than in the first five.

2. A fair die is rolled repeatedly. Calculate, correct to at least two decimal places:

a) the chance that the first 6 appears before the tenth roll;

b) the chance that the third 6 appears on the tenth roll;

c) the chance of seeing three 6's among the first ten rolls, given that there were six 6's among the first twenty rolls;

d) the expected number of rolls until six 6's appear;

e) the expected number of rolls until all six faces appear.

3. Two fair dice are rolled independently. Let X be the maximum of the two rolls, and Y the minimum.

a) What is $P(X = x)$ for $x = 1, \dots, 6$?

b) What is $P(Y = y|X = 3)$ for $y = 1, \dots, 6$?

c) What is the joint distribution of X and Y?

d) What is $E(X + Y)$?

4. Let X and Y be independent, each uniform on $\{0, 1, \dots, 100\}$. Let $S = X + Y$. For $n = 0, \dots, 200$, find:

a) $P(S = n)$; b) $P(S \leq n)$. c) Sketch graphs of these functions of n.

5. Someone plays roulette the following way: before each spin he rolls a die, and then he bets on red as many dollars as there were spots on the die. For example, if there were 4 spots he bets \$4.
If red comes up he gets the stake back plus an amount equal to the stake. If red does not come up he loses the stake. In the example above, if red comes up he gets the stake of \$4 back plus an additional \$4. If red does not come up he loses his stake of \$4. The probability of red coming up is 18/38.

a) What is his expected gain on one spin?

b) What is the expected number of spins it will take until red comes up for the first time?

c) What is the expected number of spins it will take until the first time the person bets exactly \$4 on one spin and wins.

6. A gambler repeatedly bets 10 dollars on red at a roulette table, winning 10 dollars with probability 18/38, losing 10 dollars with probability 20/38. He starts with capital of 100 dollars, and can borrow money if necessary to keep in the game.

a) Find exact expressions for the probabilities that after 50 plays the gambler is:

i) ahead; ii) not in debt.

b) Find the mean and variance of the gambler's capital after 50 plays.

c) Use the normal approximation to estimate the probabilities in a) above.

7. Suppose an airline accepted 12 reservations for a commuter plane with 10 seats. They know that 7 reservations went to regular commuters who will show up for sure. The other 5 passengers will show up with a 50% chance, independently of each other.

a) Find the probability that the flight will be overbooked, i.e., more passengers will show up than seats are available.

b) Find the probability that there will be empty seats.

c) Let X be the number of passengers turned away. Find $E(X)$.

8. A box contains w white balls and b black balls. Balls are drawn one by one at random from the box, until b black balls have been drawn. Let X be the number of draws made. Find the distribution of X,

a) if the draws are made with replacement;

b) if the draws are made without replacement.

9. **The doubling cube.** A doubling cube is a die with faces marked $2, 4, 8, 16, 32$, and 64. Suppose two doubling cubes are rolled. Let XY be the product of the two numbers. Find a) $P(XY < 100)$; b) $P(XY < 200)$; c) $E(XY)$; d) $SD(XY)$.

10. **Matching.** Suppose each of n balls labeled 1 to n is placed in one of n boxes labeled 1 to n. Assume the n placements are made independently and uniformly at random (so each box can contain more than one ball). A match occurs at place k if ball number k falls in box k. Find:

a) the probability of a match at i and no match at j;

b) the expected number of matches.

11. Data for performances of a particular surgical operation show that two operations per thousand have resulted in the death of the patient. Let X be the number of deaths due to the next thousand operations of this kind. Which of these three numbers is the smallest and which is the largest

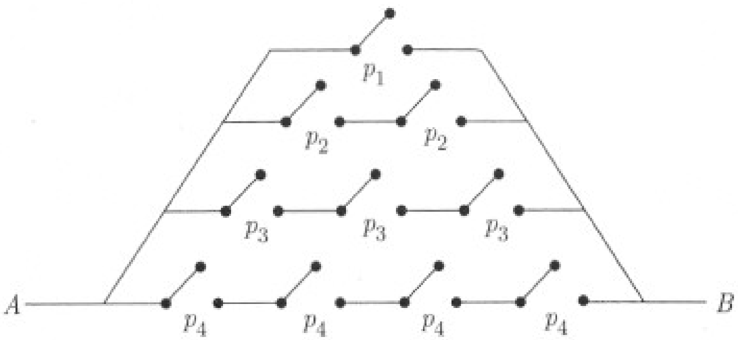
$$P(X < 2), \quad P(X = 2), \quad P(X > 2)?$$

Explain carefully the assumptions of your answer.

12. Consider an unlimited sequence of independent trials resulting in success with probability p, failure with probability q. For $s = 1, 2, \dots, f = 1, 2, \dots$ calculate the probability that s successes in a row occur before f failures in a row. [*Hint*: Let A be the event in question, $P_1 = P(A \,|\, \text{first trial a success})$, and $P_0 = P(A \,|\, \text{first trial a failure})$. Given the first trial is a success, for A to occur, either the next $s - 1$ trials must be successes, or the first failure must come at the tth trial for some $2 \leq t \leq s$, then subsequently the event A must occur starting from a failure. This gives one equation relating P_1 to P_0. Find another by conditioning on the first trial being a failure, then solve for P_0 and P_1, hence $P(A)$.]

13. Let X and Y be independent random variables with $E(X) = E(Y) = \mu$, $Var(X) = Var(Y) = \sigma^2$. Show that $Var(XY) = \sigma^2(2\mu^2 + \sigma^2)$.

14. A circuit contains 10 switches, arranged as in the figure below. Assume switches perform independently of each other, and are closed with probabilities indicated in the figure. Current flows through a switch if and only if it is closed.



a) What is the probability that current flows between points A and B?

b) Find the mean and standard deviation of the number of closed switches.

15. A roulette wheel is spun independently many times. On each spin the chance of a seven appearing is $1/38$.

a) What is the exact distribution of the number of sevens in the first 100 spins?

b) Give a simple approximation for this distribution.

c) What is the distribution of the number Z of spins required to produce three sevens?

d) What is $E(Z)$?

16. Random products mod 10. Pick two successive digits from a table of random digits from $\{0, 1, \ldots, 9\}$. Multiply them together, and let D be the last digit of this random product. For example,

$$(3, 9) \to 27 \to 7$$

$$(2, 4) \to 8 \to 8$$

Find the distribution of D, and calculate its mean.

17. Suppose N dice are rolled, where $1 \le N \le 6$.

a) Given that no two of the N dice show the same face, what is the probability that one of the dice shows a six? Give a formula in terms of N.

b) In a) the number of dice N was fixed, but now repeat assuming instead that N is random, determined as the value of another die roll. Your answer now should be simply a number, not involving N.

25. Let Y_1 and Y_2 be independent random variables each with probability distribution defined by the following table:

value	0	1	2
probability	1/2	1/3	1/6

a) Display the probability distribution of $Y_1 + Y_2$ in a table. Express all probabilities as multiples of $1/36$.

b) Calculate $E(3Y_1 + 2Y_2)$.

c) Let X_1 and X_2 be the numbers on two rolls of a fair die. Define a function f so that $(f(X_1), f(X_2))$ has the same distribution as (Y_1, Y_2).

27. A certain test is going to be repeated until done satisfactorily. Assume that repetitions of the test are independent and that each has probability 0.25 of being satisfactory. The first 5 tests cost \$100 each to perform and thereafter cost \$40 each, regardless of the outcomes. Find the expected cost of running the tests until a satisfactory result is obtained.

3.rev.1. a) $1-(5/6)^{10}$ b) $10/6$ c) 35 d) $\frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$ e) $\frac{1}{2}\left(1-\sum_{k=0}^{5}\left[\binom{5}{k}(1/6)^k(5/6)^{5-k}\right]^2\right)$

3.rev.3. a) $(2x-1)/36$ b) $2/5$ for $y = 1, 2$ and $1/5$ for $y = 3$.
c) $2/36$ for $1 \le y < x \le 6$ and $1/36$ for $y = x$ d) 7

3.rev.5. a) -18.4 cents b) 2.111 c) 12.667

3.rev.7. a) 0.1875 b) 0.5 c) 0.219

3.rev.9. a) 5/12 b) 7/12 c) 441 d) Approximately 796

3.rev.11. $P(X < 2)$ is largest, $P(X > 2)$ is smallest.

3.rev.15. a) Binomial(100, 1/38) b) Poisson(100/38) c) Negative binomial (3, 1/38) shifted to $\{3, 4, \ldots\}$ d) 3×38

3.rev.17. a) $N/6$ b) 0.3604

3.rev.19. a) $e^{-p\mu}$ b) 0.6065

3.rev.21. Negative binomial distribution on $\{0, 1, \ldots\}$ with parameters $r = 3$ and p

3.rev.23. a) $\frac{2^k (n)_k}{(2n)_k}$ b) $H/\sqrt{2n}$ tends to the Rayleigh distribution (See section 6.3).
c) $\sqrt{\pi n}$ d) 17 or so.

3.rev.25. a) Partial answer: $P(Y_1 + Y_2 = 0) = 9/36$, $P(Y_1 + Y_2 = 1) = 12/36$, $P(Y_1 + Y_2 = 2) = 10/36$. b) 10/3

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