Review Exercises Chapter 4 Continuous Distributions

1. Suppose atoms of a given kind have an exponentially distributed lifetime with rate λ. Let X_t be the number of atoms still present at time $t \geq 0$, starting from $X_0 = n$. Find formulae in terms of n, t, and λ for a) $E(X_t)$; b) $Var(X_t)$.

2. Find the constant c which makes the function $f(x) = c(x+x^2)$ for $0 < x < 1$ the density of a probability distribution on $(0,1)$. Find the corresponding c.d.f. $F(x)$. Sketch the graphs of $f(x)$ and $F(x)$. Find the expectation μ and standard deviation σ of a random variable X with this distribution. Mark the points μ, $\mu + \sigma$ on your graphs.

3. Let Y_1, Y_2, and Y_3 be three points chosen independently and uniformly from $(0,1)$, and let X be the rightmost (largest) point. Find the c.d.f., density function, and expectation of X.

4. Let X be a random variable with density $f(x) = 0.5e^{-|x|}$ $(-\infty < x < \infty)$. Find:
   a) $P(X < 1)$; b) $E(X)$ and $SD(X)$; c) the c.d.f. of X^2.

5. An ambulance station, 30 miles from one end of a 100-mile road, services accidents along the whole road. Suppose accidents occur with uniform distribution along the road, and the ambulance can travel at 60 miles an hour. Let T minutes be the response time (between when accident occurs and when ambulance arrives).
   a) Find $P(T > 30)$.
   b) Find $P(T > t)$ as a function of t. Sketch its graph.
   c) Calculate the density function of T.
   d) Calculate the mean and standard deviation of T.
   e) What would be a better place for the station? Explain.

6. Electrical components of a particular type have exponentially distributed lifetimes with mean 48 hours. In one application the component is replaced by a new one if it fails before 48 hours, and in case it survives 48 hours it is replaced by a new one anyway. Let T represent the potential lifetime of a component in continuous use, and U the time of such a component in use with the above replacement policy. Sketch the graphs of:
   a) the c.d.f. of T; b) the c.d.f. of U. Is U discrete, continuous, or neither?
   c) Find $E(U)$. [*Hint*: Express U as a function of T.]
   d) Does the replacement policy serve any good purpose? Explain.

7. **Two-sided exponential distribution.** Suppose X with range $(-\infty, \infty)$ has density $f(x) = \alpha e^{-\beta|x|}$ where α and β are positive constants.
   a) Express α in terms of β. b) Find $E(X)$ and $Var(X)$ in terms of β.
   c) Find $P(|X| > y)$ in terms of y and β. d) Find $P(X \leq x)$ in terms of x and β.

Exercises 5.1

1. Let (X, Y) have uniform distribution on the set

$$\{(x,y) : 0 < x < 2 \text{ and } 0 < y < 4 \text{ and } x < y\}.$$

Find: a) $P(X < 1)$; b) $P(Y < X^2)$.

2. A metal rod is l inches long. Measurements on the length of this rod are equal to l plus random error. Assume that the errors are uniformly distributed over the range -0.1 inch to $+0.1$ inch, and are independent of each other.
   a) Find the chance that a measurement is less than $1/100$ of an inch away from l.
   b) Find the chance that two measurements are less than $1/100$ of an inch away from each other.

3. Suppose X and Y are independent and uniformly distributed on the unit interval $(0,1)$. Find:

$$P(Y \geq \frac{1}{2} | Y \geq 1 - 2X).$$

4. Let X and Y be independent random variables each uniformly distributed on $(0,1)$. Find:
   a) $P(|X - Y| \leq 0.25)$; b) $P(|X/Y - 1| \leq 0.25)$; c) $P(Y \geq X | Y \geq 0.25)$.

5. A very large group of students takes a test. Each of them is told his or her percentile rank among all students taking the test.
   a) If a student is picked at random from all students taking the test, what is the probability that the student's percentile rank is over 90%?
   b) If two students are picked independently at random, what is the probability that their percentile ranks differ by more than 10%?

6. A group of 10 people agree to meet for lunch at a cafe between 12 noon and 12:15 P.M. Assume that each person arrives at the cafe at a time uniformly distributed between noon and 12:15 P.M., and that the arrival times are independent of each other.
   a) Jack and Jill are two members of the group. Find the probability that Jack arrives at least two minutes before Jill.
   b) Find the probability of the event that the first of the 10 persons to arrive does so by 12:05 P.M., and the last person arrives after 12:10 P.M.

7. Let X and Y be two independent uniform $(0,1)$ random variables. Let M be the smaller of X and Y. Let $0 < x < 1$.
   a) Represent the event $(M \geq x)$ as the region in the plane, and find $P(M \geq x)$ as the area of this region.
   b) Use your result in a) to find the c.d.f. and density of M. Sketch the graph of these functions.

8. Let $U_{(1)}, \ldots, U_{(n)}$ be the values of n independent uniform $(0,1)$ random variables arranged in increasing order. Let $0 \leq x < y \leq 1$.
   a) Find and justify a simple formula for $P(U_{(1)} > x \text{ and } U_{(n)} < y)$.
   b) Find a formula for $P(U_{(1)} \leq x \text{ and } U_{(n)} < y)$.

9. **A triangle problem.** Suppose a straight stick is broken in three at two points chosen independently at random along its length. What is the chance that the three sticks so formed can be made into the sides of a triangle?

Exercises 5.2

1. Suppose that (X, Y) is uniformly distributed over the region $\{(x, y) : 0 < |y| < x < 1\}$. Find:

a) the joint density of (X, Y); b) the marginal densities $f_X(x)$ and $f_Y(y)$.

c) Are X and Y independent? d) Find $E(X)$ and $E(Y)$.

2. Repeat Exercise 1 for (X, Y) with uniform distribution over $\{(x, y) : 0 < |x| + |y| < 1\}$.

3. A random point (X, Y) in the unit square has joint density $f(x, y) = c(x^2 + 4xy)$ for $0 < x < 1$ and $0 < y < 1$, for some constant c.

a) Evaluate c. b) Find $P(X \le a)$, $0 < a < 1$. c) Find $P(Y \le b)$, $0 < b < 1$.

4. For random variables X and Y with joint density function

$$f(x, y) = 6e^{-2x-3y} \qquad (x, y > 0)$$

and $f(x, y) = 0$ otherwise, find:

a) $P(X \le x, Y \le y)$; b) $f_X(x)$; c) $f_Y(y)$.

d) Are X and Y independent? Give a reason for your answer.

5. Let X be exponentially distributed with rate λ, independent of Y, which is exponentially distributed with rate μ. Find $P(X \ge 3Y)$.

6. Let X and Y have joint density

$$f(x, y) = \begin{cases} 90(y - x)^8 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find $P(Y > 2X)$. b) Find the marginal density of X.

c) Fill in the blanks (explain briefly):
The joint density f above is the joint density of the _____ and _____ of ten independent uniform (0, 1) random variables.

7. Two points are picked independently and uniformly at random from the region inside a circle. Let R_1 and R_2 be the distances of these points from the center of the circle. Find $P(R_2 \le R_1/2)$.

4.rev.1. a) $ne^{-\lambda t}$ b) $ne^{-\lambda t}(1 - e^{-\lambda t})$

4.rev.3. Density: $3x^2$ if $0 < x < 1$. Expectation: $3/4$

4.rev.5. a) 0.4 b) If $0 < t \le 30$, then $P(T > t) = \frac{100-2t}{100}$; If $30 < t \le 70$, then $P(T > t) = \frac{70-t}{100}$. c) $f_T(t) = 2/100$ if $0 < t \le 30$, $= 1/100$ if $30 < t \le 70$. d) mean 29, SD 19.8 e) Locate the station at the midpoint of the road.

4.rev.7. a) $\alpha = \frac{\beta}{2}$ b) $E(X) = 0$, $Var(X) = 2/\beta^2$ c) $e^{-\beta y}$ if $y > 0$ d) $1 - (1/2)e^{-\beta x}$ if $x > 0$; $(1/2)e^{\beta x}$ if $x < 0$

5.1.1. a) 7/12 b) 5/36

5.1.3. 7/12

5.1.5. a) 0.1 b) 0.81

5.1.7. a) $(1-x)^2$ b) If $0 < x < 1$ then $P(M \le x) = 1-(1-x)^2$ and $f_M(x) = 2(1-x)$

5.1.9. 1/4

5.2.1. a) If $0 < |y| < x < 1$ then $f_{X,Y}(x, y) = 1$ if $0 < y < 1$ then $f_Y(y) = 1 - |y|$ b) If $0 < x < 1$ then $f_X(x) = 2x$; c) no d) $E(X) = 2/3$, $E(Y) = 0$

5.2.3. a) 3/4 b) $\frac{3}{4}\left(\frac{a^3}{3} + a^2\right)$ c) $\frac{3}{4}\left(\frac{b}{3} + b^2\right)$

5.2.5. $\frac{\mu}{3\lambda+\mu}$

5.2.7. 1/8