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Name: Course Assistant Fall '03

Math 21a Final Exam – Thursday, May 16th, 2002

Please circle your section: ***Note this version is just for the Regular and Physics Sections***

Christophe Cornut David Shim (CA) MWF 10-11	Spiro Karigiannis Michael Simonetti (CA) MWF 10-11	Spiro Karigiannis Gloria Hou (CA) MWF 11-12
Andy Engelward Nathan Moore (CA) T/Th 10-11:30	Andy Engelward Alexey Gorshkov (CA) T/Th 11:30-12	

Question	Points	Score
1	8	
2	8	
3	8	
4	16	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

#10 is on partial differential equation which are no longer covered in this course

You have three hours to take this final exam. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order - you should move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers will receive little or no credit! Please show all of your work and be sure to write neatly - illegible answers will also receive little or no credit.

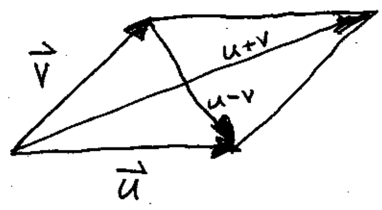
If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one page (standard 8 and a half by 11 inch) of notes during the test, but you are not allowed to use any other references or calculators during this test.

Good luck! Focus and do well!

Question 1. (8 points total)

(a) Suppose that the diagonals of a certain parallelogram are perpendicular to each other. By considering the diagonals of the parallelogram as vectors, and taking their dot product, determine the relationship between the lengths of the sides of the parallelogram.



$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$ (since perpendicular)

$\vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{u}\cdot\vec{v}} - \vec{v}\cdot\vec{v} = 0$

$\vec{u}\cdot\vec{u} = \vec{v}\cdot\vec{v}$

$|\vec{u}|^2 = |\vec{v}|^2$

$|\vec{u}| = |\vec{v}|$

The sides have equal length.
Rhombus!

$\boxed{\text{This question tests your knowledge of vectors (9.2) and the dot product (9.3)}}$

(b) What is the relationship between the cross product of the two diagonals, again considered as vectors, and the area of the parallelogram described in part (a)? Does this relationship hold for any given parallelogram, not just one whose diagonals are perpendicular?

Cross product of the diagonals $(u+v)\times(u-v)$

(Recall: $\vec{a}\times\vec{a} = \vec{0}$, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$)

$= \cancel{u\times u} - \underbrace{(u\times v)}_{=-v\times u} + (v\times u) - \cancel{v\times v}$

$= 2(v\times u)$

Recall Area $= |v\times u|$

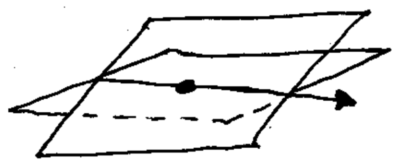
Therefore the magnitude of the cross product is twice the area.

We also did not have to specify the vector were $\perp$ so this holds true for all parallelograms!

$\boxed{\text{This question tests your knowledge of the cross product (9.4)}}$

Question 2. (8 points total)

(a) Give a set of parametric equations for the line that lies in the planes $2x+3y-z=2$ and $-x+2y+2z=4$.



Remember, to define a line you need a point on it and a direction.

This question tests your knowledge of lines and planes (9.5)

Point: take $x=0$

$$\begin{cases} 3y - z = 2 \\ 2y + 2z = 4 \end{cases} \Rightarrow \begin{matrix} y=1 \\ z=1 \end{matrix}$$

So, $\vec{P}=\langle 0,1,1\rangle$ lies on the line.

Direction is $\perp$ to both normals so

$$\vec{v} = (2,3,-1)\times(-1,2,2)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 2 & 2 \end{vmatrix} = \langle 8,-3,7\rangle$$

So $\vec{r}(t)=\vec{P}+t\vec{v}$

$$\boxed{\begin{aligned} x &= 8t \\ y &= 1-3t \\ z &= 1+7t \end{aligned}}$$

(b) What is the angle between the two planes in part (a)? (recall here angle means the acute angle between the planes' normal vectors)

The angle between the planes is equal to the angle between their normals

$\sin\theta$ ← correction

$$|\vec{n}_1 \times \vec{n}_2| = |\vec{n}_1||\vec{n}_2| \cancel{\cos\theta}$$

$$\sqrt{122} = 3\sqrt{14}\ \overset{\sin}{\cancel{\cos\theta}}$$

$$\theta = \underset{\sin^{-1}}{\cancel{\cos^{-1}}}\left(\frac{\sqrt{122}}{3\sqrt{14}}\right)$$

Question 3. (8 points total)

(a) Suppose that at the point (4, 5, 6) the function $f(x,y,z)$ increases most rapidly in the direction $\langle 4,0,-3\rangle$ and that the rate of increase of $f(x,y,z)$ in this direction is equal to 7. What is the rate of increase of $f(x,y,z)$ at the point (4, 5, 6) in the direction $\langle -1,2,-2\rangle$?

at (4,5,6) the gradient is in direction <4,0,-3> w/ magnitude 7

so $\nabla f = 7 \cdot \frac{\langle 4,0,-3\rangle}{5} = \langle \frac{28}{5}, 0, \frac{-21}{5}\rangle$

Directional derivative

$$D_{\vec{u}} f = \nabla f \cdot u$$
$$= \langle \tfrac{28}{5}, 0, \tfrac{-21}{5}\rangle \cdot \frac{\langle -1,2,-2\rangle}{3}$$
$$\boxed{= \cancel{\langle} \tfrac{-28}{15} + 0 + \tfrac{14}{5} \cancel{\rangle}} = \frac{14}{15}$$

correction

Section 11.6

(b) Let S be the level surface of $f(x,y,z)$ that goes through the point (4, 5, 6) (where $f(x,y,z)$ is the same function as in part (a)). Write down an equation for the plane that is tangent to S at the point (4, 5, 6).

$\cancel{L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}$

The gradient must be normal to the surface S

$$\frac{28}{5}x - \frac{21}{5}z = d$$
$$\frac{28}{5}(4) - \frac{21}{5}(6) = \frac{\cancel{238}\ -14}{5}$$

$$\boxed{28x - 21z = \cancel{238}\ -14}$$

Section 11.4 $\quad 4x - 3z = -2$

Question 4. (16 points total)

Consider the function $f(x,y) = x^2 + y^2 + 2y - 1$

(a) Find and classify all critical points of $f(x,y)$ in the plane.

How to find critical points: ① ~~set~~ $\nabla f = 0$
② solve for x and y (and z)

$\nabla f = (2x, 2+2y) = (0,0)$

$\begin{cases} 2x = 0 \\ 2+2y = 0 \end{cases} \longrightarrow \boxed{\begin{matrix} x=0 \\ y=-1 \end{matrix}}$ minimum

$D = 2 \cdot 2 + 0 = 4$ (positive means either max or min

$f_{xx} = 2$ (positive means minimum)

Section 11.7

(b) Consider restricting $f(x,y)$ to the constraint curve $y = x^2 + 1$. Find the extreme points of $f(x,y)$ on this curve using the Lagrange multiplier method. You don't need to classify them as minimums or maximums at this point. Note you might want to do part (d) before parts (b) and (c).

Lagrange! Section 11.8

① Set up system of equations

$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$ (where f is fcn to optimize and g is the restraint)

② solve for critical points

$g(x,y) = x^2 - y + 1 = 0$

$\nabla f = (2x, 2+2y)$

$\nabla g = (2x, -1)$

$\begin{cases} 2x = 2x \cdot \lambda \\ 2+2y = -1 \cdot \lambda \\ x^2 - y + 1 = 0 \end{cases} \Rightarrow \boxed{\begin{matrix} x=0 \\ y=1 \end{matrix}}$

$\lambda = -4$

Question 4 continued

(c) Still restricting $f(x,y)=x^2+y^2+2y-1$ to the constraint curve $y=x^2+1$, now find any extreme points by parametrizing the curve (in doing this you can check your work in part (b)). Again, you can wait to classify them until you do part (d).

Optimization under constraint w/o Lagrange (See Sec 11.7 Ex 6)

- Use restraint to eliminate a varible in function
- Solve for points

$\vec{r}(x) = \langle x, x^2+1 \rangle$

$f(x, x^2+1) = x^2+(x^2+1)^2 + 2(x^2+1)-1$

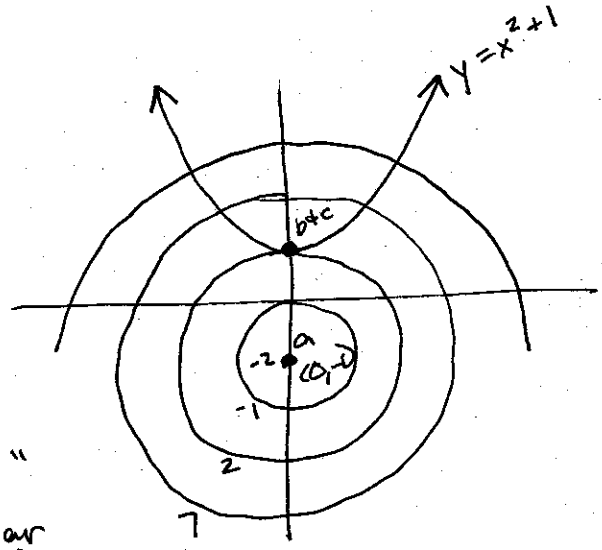
$\frac{\partial f}{\partial x} = 10x + 4x^3 = 0$

$\begin{cases} x=0 \\ y=1 \end{cases}$ $\boxed{\langle 0,1 \rangle}$

(d) Now sketch the constraint curve and several of the level curves of $f(x,y)$, explain your answers to parts (a), (b) and (c) using the picture, and finally classify any extreme points you found in (b) and (c) as either maximums or minimums.

$f(x,y) = x^2+y^2+2y-1$

$= x^2+(y+1)^2-2$

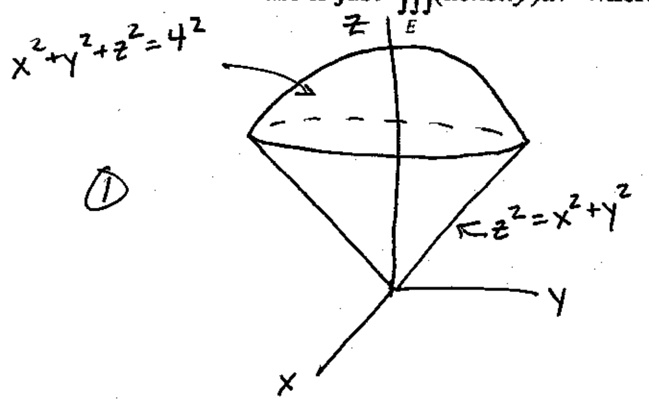


a) part a found the global minimum for the function (a parabola)
note the "bottom of the valley"

b+c) Both Lagrange and regular optimization found a (global) minimum at (0,1)
note that the constrait is parallel to the level curve!

Question 5. (10 points total)

It's hot, and this time you're not in a "dessert," but you are in fact eating one. Suppose you are eating an ice cream cone, whose shape (including both the cone and the ice cream in it) can be described as the region above the surface $z^2 = x^2 + y^2$ and below a sphere of radius 4 centered at the origin (if you sketch this you'll see that it's a pretty fat cone!) Suppose the density of the ice cream (along with the cone) varies depending on the distance d from the bottom tip of the cone, given by $\sigma = k(10 - d)$, where k is a constant. Find the mass of the ice cream cone (note that mass is just $\iiint_E (density)dV$ where E is the region described above).

① 

Triple integrals (Sec 12.7)

How to attack these:

① DRAW THE SHAPE
This will be the easiest way to ...

② Pick a coordinate system
Probably the one it looks like

③ Find the proper limits
This step takes some practice

④ Integrate

② ~~Since~~ both the sphere and the cone are easily defined by spherical coordinates $\rho = 4$, $\phi = \frac{\pi}{4}$

③ $\begin{cases} 0 \le \rho \le 4 & \text{(from the origin to the top of the cone)} \\ 0 \le \phi \le \frac{\pi}{4} & \text{(from the z-axis down to the cone)} \\ 0 \le \theta \le 2\pi & \text{(all around the z-axis)} \end{cases}$

correction

④ $\sigma = k(10-d) = k(10 - \cancel{z}) = k(10 - \rho \cancel{\cos\phi})$

$$\iiint_E \sigma \, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 k(10 - \rho \cancel{\cos\phi})\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{64}{3} k(10 - 3\cancel{\cos\phi})\sin\phi \, d\phi \, d\theta$$

$$\boxed{= \cancel{\frac{-32}{3}(20\sqrt{2} - 37)k\pi}} = \frac{448}{3}(2 - \sqrt{2})k\pi$$

Question 6. (10 points total)

(a) Let $f(x,y,z)$ be a potential function for a conservative vector field **F**, i.e. $\mathbf{F} = \nabla f$. Consider a level surface M for the function $f(x,y,z)$, so that M is given by $f(x,y,z) = k$, for some constant k (such a surface is called an equipotential surface). If C is a curve (not necessarily closed) on such a surface M, then explain why it is that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$.

The line integral plus the description of the surface might fool you into thinking Stokes (it got me) but the curve is not closed so we use our other tool, the fundemental theorem of line integrals (13.3)

$$\int_C F\cdot dr$$

$$\cancel{\iint_M \text{curl}\, F\cdot dS}$$

$$\cancel{\iint_M (0)\, dS} = 0$$

$$\int_C \nabla f \cdot dr = f(\vec{r}(b) - \vec{r}(a))$$

but since boths the beginning and end point lie on a level surface

$$\vec{F}(b) = \vec{r}(a)$$

correction!

$$\Longrightarrow \int_C F\cdot dr \boxed{= 0}$$

(b) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle -e^y\sin(x), e^y\cos(x)\rangle$ and C is the curve given by

$\mathbf{r}(t) = \langle \pi - \pi\cos(t), \pi\sin(t)\rangle$ with $0 \le t \le \pi$

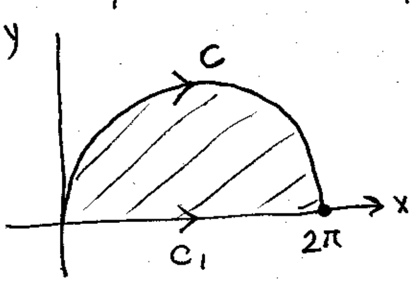
Let's try this as a normal line integral and see what happens.

$$\int_C F\cdot dr = \int_a^b F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$r'(t) = \langle \pi\sin t, \pi\cos t\rangle$$

$$F(\vec{r}(t)) = \langle -e^{\pi\sin t}\sin(\pi - \pi\cos t), e^{\pi\sin t}\cos(\pi - \pi\cos t)\rangle$$

yuk! Let's apply Green's (13.4) and see what happens



$$\int_{C_1} F\cdot dr - \int_C F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{C_1} F\cdot dr \qquad r = \langle x, 0\rangle$$

$$= \int_0^{2\pi} \langle -\sin x, \cos x\rangle\cdot\langle 1, 0\rangle\, dx = 0$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -e^y\sin x + e^y\sin x = 0$$

$$\int_C F\cdot dr = 0 - 0 \boxed{= 0}$$

Question 7. (10 points total)

Use Green's Theorem and the vector field $\mathbf{F} = \langle 0, x^3 y \rangle$ to compute the double integral $\iint_R (3x^2 y)\,dA$ where R is the region inside the ellipse $x^2 + \frac{y^2}{4} = 1$

For the boundary of the ellipse note that $(\cos(t))^2 + \frac{(2\sin(t))^2}{4} = 1$

Well, Green's says $\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, but how does this relete to this question?

for $\vec{F}$ $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 3x^2 y$

so $\iint_R (3x^2 y)\,dA = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C F \cdot dr$

Now, How to set up line integral

1. parametrize the curve and find limits
2. Plug into $\int_a^b F(\vec{r}(t)) \cdot r'(t)\,dt$
3. Integrate

1. $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$ (the hint should help you here) $\quad 0 \leq t \leq 2\pi$

$\vec{r}'(t) = \langle -\sin t, 2\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle 0, \cos^3 t \cdot 2\sin t \rangle$

2.

3. $\int_0^{2\pi} 4\cos^4 t \sin t\,dt$ $\quad u = \cos t$, $du = -\sin t\,dt$

$= 4 \cdot \frac{-1}{5} \cos^5 t \Big|_0^{2\pi} = \boxed{0}$

Question 8. (10 points total)

Let S be the surface given by $z = x^2 - y^2$. Let C be the curve on the surface S where $x^2 + y^2 = 1$, oriented counter-clockwise as one looks down the z-axis. Use Stokes' Theorem to calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field $\langle x^2 + z^2, y, z \rangle$

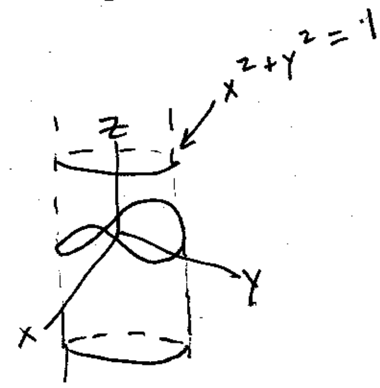
Direct application of Stokes' (13.7)

$\oint F \cdot dr = \iint_S \text{curl } F \cdot dS$

$= \iint_S \text{curl } F \cdot (r_u \times r_v)\, du\, dv$

1. Draw surface
2. Find curl F
3. Parametrize surface ~~and find limits~~
4. Find curl $F \cdot (r_u \times r_v)$
5. Integrate over proper limits

1



2 $\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+z^2 & y & z \end{vmatrix} = \langle 0, 2z, 0 \rangle$

3 $\vec{r}(x,y) = \langle x, y, x^2 - y^2 \rangle$

4 $r_x = \langle 1, 0, 2x \rangle$

$r_y = \langle 0, 1, -2y \rangle$

$r_x \times r_y = \langle -2x, +2y, 1 \rangle$

$\text{curl } F \cdot (r_x \times r_y) = 0 + 2yz + 0$

5 $\iint_S (+2y(x^2 - y^2))\, dA$

$x = r\cos\theta$
$y = r\sin\theta$

$= \int_0^{2\pi} \int_0^1 +2\, r\cos\theta\, (r^2 \cos 2\theta)\, dr\, d\theta$

$= \underbrace{\int_0^{2\pi} \cos\theta \cos 2\theta\, d\theta}_{=0} \int_0^1 +2r^3\, dr$

$\boxed{=0}$

Question 9. (10 points total)

In appropriate units, the charge density $\sigma(x,y,z)$ in a region in space is given by $\sigma = \nabla\cdot\mathbf{E} = \mathrm{div}(\mathbf{E})$ where $\mathbf{E}$ is the electric field. *(Note, you don't need to know any physics to answer this problem!)* Consider the unit cube located at the origin (the region given by $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$). What is the total charge in this cube if $\mathbf{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(xyz), z(1-z)e^{xyz}\rangle$? (note, the total charge in a region is the integral of the charge density over the region)

Be sure to show all your work and explain your reasoning to receive full credit for your answer.

$\iiint_E \mathrm{div}(E)\,dV$

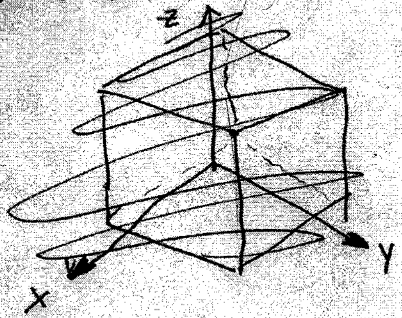
First find $\mathrm{div}(E) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$\mathrm{div}(E) = \frac{(1-x)xyz}{1+xyz} + (1-x)\log(1+xyz) - x\log(1+xyz)$

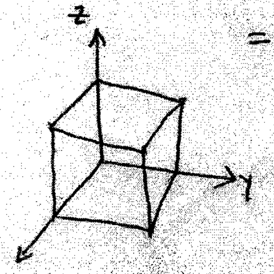
$+ x(1-y)yz\sec^2(1+xyz) + (1-y)\tan(1+xyz) - y\tan(1+xyz)$

$+ \ldots$ (13.8)

Ugh, impossible to integrate, Lets try Div Thm even though we'll have to break it into 6 surfaces!



$\iiint_E \mathrm{div}(E)\,dV = \oiint_S F\cdot dS$



$= \iint_{S_1} F\cdot dS$

$+ \iint_{S_2} F\cdot dS$

$+ \cdots$

$+ \iint_{S_6} F\cdot dS$

surface

$x=0$, $\vec{n} = \langle -1,0,0\rangle$, $F\cdot dS = \langle 0,?,?\rangle\cdot\langle -1,0,0\rangle = 0 \rightarrow \iint_S F\,dS = 0$

$x=1$, $\vec{n} = \langle 1,0,0\rangle$, $F\cdot dS = 0 \rightarrow \iint_S F\cdot dS = 0$

$y=0$, $\vec{n} = \langle 0,-1,0\rangle$, " $\longrightarrow = 0$

$y=1$, $\vec{n} = \langle 0,1,0\rangle$, " $\longrightarrow = 0$

$z=0$, $\vec{n} = \langle 0,0,-1\rangle$, " $\longrightarrow = 0$

$z=1$, $\vec{n} = \langle 0,0,1\rangle$, " $\longrightarrow = 0$

$\iiint_E \mathrm{div}(E)\,dV = \oiint_S F\cdot dS \boxed{= 0}$