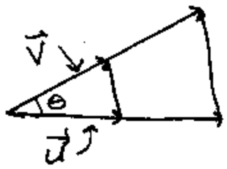
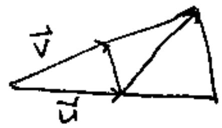
Math 21a Final Exam <u>Answers</u> Tues, May 22 2001

(1)



let $\vec{u}, \vec{v}$ be half the length of the two sides, so $|\vec{u}| = |\vec{v}|$ since its isoceles,

then



one diagonal is $2\vec{v} - \vec{u}$, and the other is $2\vec{u} - \vec{v}$

since the diagonals are perpendicular then

$$(2\vec{v} - \vec{u}) \cdot (2\vec{u} - \vec{v}) = 0, \quad \text{so} \quad 4(\vec{v}\cdot\vec{u}) - 2(\vec{v}\cdot\vec{v}) - 2(\vec{u}\cdot\vec{u}) + (\vec{u}\cdot\vec{v}) = 0,$$

$$\text{so} \quad 5(\vec{v}\cdot\vec{u}) = 2\left(|\vec{v}|^2 + |\vec{u}|^2\right)$$

$$\text{and so} \quad \vec{v}\cdot\vec{u} = \frac{2}{5}\left(|\vec{v}|^2 + |\vec{u}|^2\right) = |\vec{u}||\vec{v}|\cos\theta$$

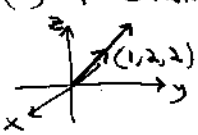
$$\text{so} \quad \cos\theta = \frac{2}{5}\left(\frac{|\vec{v}|^2 + |\vec{u}|^2}{|\vec{u}||\vec{v}|}\right) \quad \text{but } |\vec{u}| = |\vec{v}|, \text{ so } \cos\theta = \frac{2}{5}(2) = \frac{4}{5}$$

(2) (a) $\nabla F(1,2,2) \ldots \langle z^3 + 2xy,\ z^2 + x^2,\ 3xz^2 + 2zy \rangle$ when $(x,y,z) = (1,2,2)$

$$= \langle 12, 5, 20 \rangle$$

(gradient vector is perpendicular to the level surface)

(b) radially outward from $(1,2,2)$ is in direction $\langle 1,2,2 \rangle$



so unit vector in this direction $= \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ (since $\langle 1,2,2 \rangle$ has length 3)

directional derivative is just $\left(\nabla F(1,2,2)\right) \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

$$= \langle 12, 5, 20 \rangle \cdot \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle = \frac{62}{3}$$

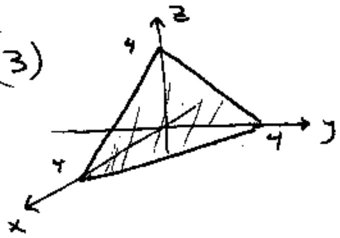
(c) implicit differentiation (not on Spring 2002 final):

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \text{at } (1,2,2) = \frac{-12}{20} = \frac{-3}{5}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}, \quad \text{at } (1,2,2) = \frac{-5}{20} = -\frac{1}{4}$$

(d) $z(x,y) = 2 - \frac{3}{5}(x-1) - \frac{1}{4}(y-2)$,

so $z(1.1, 2.04) = 2 - \frac{3}{5}(0.1) - \frac{1}{4}(0.04) = 1.93$

(3) 

note along the boundaries and at vertices of this triangle $f(x,y,z) = xy^3z^4$ equals $0$ (since at least one of $x,y,z$ is $0$ there).

Check interior of triangle:

have constraint $x+y+z=4$, and want to maximize $f(x,y,z) = xy^3z^4 \implies$ Lagrange

$\nabla f = \lambda \nabla g$ where $g(x,y,z) = x+y+z$

so $\langle y^3z^4, 3xy^2z^4, 4xy^3z^3 \rangle = \lambda \langle 1,1,1 \rangle$

so $\left.\begin{array}{l} y^3z^4 = \lambda \\ 3xy^2z^4 = \lambda \\ 4xy^3z^3 = \lambda \end{array}\right\}$ so $y^3z^4 = 3xy^2z^4 \implies y = 3x$ (since $x,y,z \neq 0$)

$4xy^3z^3 = y^3z^4 \implies 4x = z$

and $x+y+z=4$

so now $y=3x$, $z=4x$, $x+y+z=4$

so $x+(3x)+(4x) = 8x = 4$

so $x = \frac{1}{2}$, $y = 3x = \frac{3}{2}$, $z = 4x = 2$

check at this point $f(x,y,z) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3(2)^4 = 27$

which must be the maximum (greater than value on boundaries, vertices)

(4) Maximize $f(s,p) = s^{0.7}p^{0.3}$ on $g(s,p) = 7s+12p = 120$

so $\nabla f = \langle 0.7\, s^{-0.3}p^{0.3},\ 0.3\, s^{0.7}p^{-0.7} \rangle = \lambda \langle 7, 12 \rangle$

(note $s$ or $p=0$ is clearly not a maximum for $f(s,p)$)

$0.7\, s^{-0.3}p^{0.3} = 7\lambda \qquad 0.3\, s^{0.7}p^{-0.7} = 12\lambda$

$\lambda = 0.1\, s^{-0.3}p^{0.3} = 0.025\, s^{0.7}p^{-0.7}$

so $\dfrac{0.1}{0.025} = 4 = \dfrac{s^{0.7}p^{-0.7}}{s^{-0.3}p^{0.3}} = \dfrac{s}{p} \qquad s = 4p$

so $7s+12p = 7(4p)+12p = 120 \implies p = 3$,

then $s = 12$

and so then $f(12,3) = 12^{.7}3^{.3} > 0$

gives us a maximum value

(5) (a) if $x = r\cos\theta \quad y = r\sin\theta$ then $\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta}$

$$= -\frac{\partial T}{\partial x}(r\sin\theta) + \frac{\partial T}{\partial y}(r\cos\theta)$$

(b) since $\frac{\partial U}{\partial r} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial r}$ where now $U = \frac{\partial T}{\partial x}$

then $\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\frac{\partial y}{\partial r}$

$$= \frac{\partial^2 T}{\partial x^2}(\cos\theta) + \frac{\partial^2 T}{\partial x \partial y}(\sin\theta)$$

(c) $\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} = \left(\frac{\partial T}{\partial x}\right)\cos\theta + \left(\frac{\partial T}{\partial y}\right)(\sin\theta)$ (B)

now from (b) we know $\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)$, and can find

$$\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial^2 T}{\partial y^2}\sin\theta + \frac{\partial^2 T}{\partial x \partial y}\cos\theta$$

Note- almost no one got this problem right all the way through!

so $\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right) = \frac{\partial^2 T}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial y}\right)\sin\theta$

$$= \left(\frac{\partial^2 T}{\partial x^2}\cos\theta + \frac{\partial^2 T}{\partial x \partial y}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 T}{\partial y^2}\sin\theta + \frac{\partial^2 T}{\partial x \partial y}\cos\theta\right)\sin\theta$$

$$= 2\sin\theta\cos\theta\frac{\partial^2 T}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 T}{\partial x^2} + \sin^2\theta\frac{\partial^2 T}{\partial y^2} \quad \text{(A)}$$

we know $\frac{\partial T}{\partial \theta}$ from (a) and so $\frac{\partial^2 T}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-\frac{\partial T}{\partial x}(r\sin\theta) + \frac{\partial T}{\partial y}(r\cos\theta)\right)$

and $\frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\frac{\partial y}{\partial \theta} = \frac{\partial^2 T}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 T}{\partial x \partial y}(r\cos\theta)$

and $\frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)\frac{\partial y}{\partial \theta} = \frac{\partial^2 T}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 T}{\partial y^2}(r\cos\theta)$

so $\frac{\partial^2 T}{\partial \theta^2} = -\left(\frac{\partial^2 T}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 T}{\partial x \partial y}(r\cos\theta)\right)(r\sin\theta) + (r\cos\theta)\left(-\frac{\partial T}{\partial x}\right)$

$$(-r\sin\theta)\left(\frac{\partial T}{\partial y}\right) + \left(\frac{\partial^2 T}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 T}{\partial y^2}(r\cos\theta)\right)(r\cos\theta)$$

(C) $= \frac{\partial^2 T}{\partial x^2}(r^2\sin^2\theta) - 2r^2\cos\theta\sin\theta\left(\frac{\partial^2 T}{\partial x \partial y}\right) + r^2\cos^2\theta\frac{\partial^2 T}{\partial y^2}$ $-r\left(\cos\theta\frac{\partial T}{\partial x} + \sin\theta\frac{\partial T}{\partial y}\right)$

holy cow!

so $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \text{(A)} + \frac{1}{r}\text{(B)} + \frac{1}{r^2}\text{(C)}$

$$= \frac{\partial^2 T}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 T}{\partial y^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial^2 T}{\partial x \partial y}(2\sin\theta\cos\theta - 2\sin\theta\cos\theta)$$

$$+ \frac{1}{r}\left((\cos\theta)\frac{\partial T}{\partial x} + (\sin\theta)\frac{\partial T}{\partial y}\right) - \frac{1}{r}\left(\cos\theta\frac{\partial T}{\partial x} + \sin\theta\frac{\partial T}{\partial y}\right) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(6) (This change of coordinates technique was not covered spring 2002, so this type of question will not be on 2002 spring final)

(a) $y+2x=u \qquad u(x,y)=y+2x \qquad y+2x=u$

$y-x=v \qquad v(x,y)=y-x \qquad 2y-2x=2v$

$3y = u+2v \quad y=\frac{u+2v}{3}$

similarly $x=\frac{u-v}{3}$

(b) average value over R

$$= \frac{\iint_R f(u,v)\,du\,dv}{\iint_R dA}$$

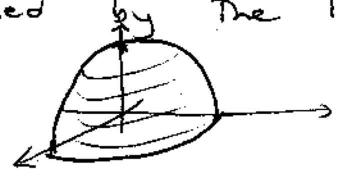
(c) $\text{stretching factor} = \dfrac{1}{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}} = \dfrac{1}{\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}} = \dfrac{1}{3}$

$$\iint_R f(u,v)\,du\,dv = \int_4^{10}\int_{-2}^{1} -\tfrac{1}{3}uv\,dv\,du = 21$$

$$\iint_R dA = \int_4^{10}\int_{-2}^{1} \tfrac{1}{3}\,du\,dv = 6$$, so average value over $R = \frac{21}{6} = \frac{7}{2}$

(7) sketch the region defined by the limits of integration:

$$\left.\begin{array}{l} 0 \le z \le \sqrt{4-r^2} \\ 0 \le r \le 2 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right\} \Longrightarrow$$



$\frac{1}{8}$ of radius 2 sphere

i.e. $0 \le \rho \le 2$

$0 \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le \pi/2$

$r\,dz\,dr\,d\theta$ becomes

$\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

left with integrand $r^2 z^2$

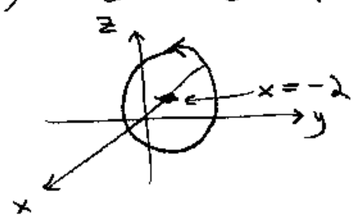
$$r^2 = \left(\frac{x}{\cos\theta}\right)^2 = \left(\frac{\rho\cos\theta\sin\phi}{\cos\theta}\right)^2 = \rho^2\sin^2\phi \qquad z^2 = \rho^2\cos^2\phi$$

integral becomes $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^6 \sin^3\phi\cos^2\phi\,d\rho\,d\phi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^7}{7}\sin^3\phi\cos^2\phi\Big|_0^2 d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \frac{128}{7}\sin\phi(1-\cos^2\phi)(\cos^2\phi)\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\left(-\frac{128}{21}\cos^3\theta + \frac{128}{35}\cos^5\theta\right)\Big|_0^{\pi/2} d\theta = \frac{64\pi}{21} - \frac{64\pi}{35}$$

(8) Stokes' Theorem says $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot dS$




where $S$ is the surface of the circle cross-section shown. When $x=-2$
$4x^2+y^2+z^2=25$ becomes $y^2+z^2=9$, i.e.
a radius 3 circle, of area $9\pi$

the outward pointing normal for $S$ relative to the path direction shown is $\langle 1,0,0\rangle$,
so $(\text{curl}\,\vec{F})\cdot dS = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dA = (x^2-x)\,dA$

(where $F = \langle P,Q,R\rangle = \langle xy-x, xz, x^2y\rangle$ is the given vector field)

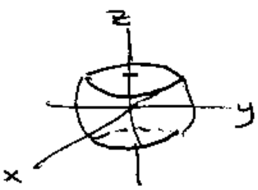
Aha! for $S$, the circular cross section shown, $x=-2$,
so this $(\text{curl}\,\vec{F})\cdot dS$ is just $((-2)^2-(-2))\,dA = 6\,dA$,

and so $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot dS = \iint_S 6\,dA = 6\iint_S dA$

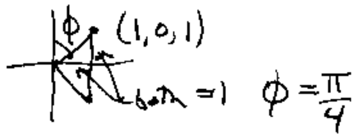
and $\iint_S dA$ is just the area of the circle, $= 9\pi$

so Stokes' Theorem gives us $6\cdot 9\pi = 54\pi$

(9) (a)




parametrize strip around sides with spherical coordinates: $\rho=\sqrt{2}$, $0\le\theta\le 2\pi$, $\frac{\pi}{4}\le\phi\le\frac{3\pi}{4}$




so parametrization is $\vec{r}(\theta,\phi)$
$= \langle \sqrt{2}\cos\theta\sin\phi, \sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\phi\rangle$

and $\vec{r}_\theta = \langle -\sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\theta\sin\phi, 0\rangle$
$\vec{r}_\phi = \langle \sqrt{2}\cos\theta\cos\phi, \sqrt{2}\sin\theta\cos\phi, -\sqrt{2}\sin\phi\rangle$

then $|\vec{r}_\theta\times\vec{r}_\phi| = \sqrt{4\sin^4\phi + 4\cos^2\phi\sin^2\phi} = 2\sin\phi$ ($\sin\phi>0$ for $\frac{\pi}{4}\le\phi\le\frac{3\pi}{4}$)

get $\int_0^{2\pi}\int_{\pi/4}^{3\pi/4} 2\sin\phi\,d\phi\,d\theta = 4\pi\sqrt{2}$

plus top and bottom (each area $\pi$)
get surface of drum $= 2\pi + 4\pi\sqrt{2}$

(9)(b) find $\iint_S \vec{F} \cdot dS$, already parametrized with
$\vec{r}(\theta, \phi) = \langle \sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, \sqrt{2}\cos\phi \rangle$
with $0 \le \theta \le 2\pi \quad \pi/4 \le \phi \le 3\pi/4$

so $\vec{F} = \langle x, y, 0 \rangle = \langle \sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, 0 \rangle$

and $(\vec{r}_\theta \times \vec{r}_\phi) = \langle 2\sin^2\phi\cos\theta, 2\sin^2\phi\sin\theta, 2\cos\phi\sin\phi \rangle$

so $$\iint_S \vec{F} \cdot dS = \int_{\pi/4}^{3\pi/4} \int_0^{2\pi} 2\sqrt{2}\left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta\right) d\theta\, d\phi$$

$$= \int_{\pi/4}^{3\pi/4} \int_0^{2\pi} 2\sqrt{2}\sin^3\phi \, d\theta\, d\phi = 4\pi\sqrt{2}\int_{\pi/4}^{3\pi/4} \sin^3\phi\, d\phi$$

since $\sin^3\phi = \sin\phi(1-\cos^2\phi)$

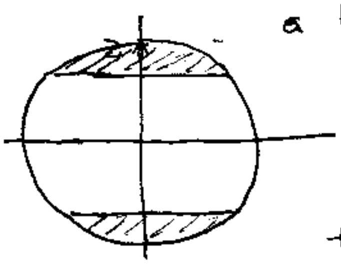
then get $$4\pi\sqrt{2}\left(\int_{\pi/4}^{3\pi/4} \sin\phi\, d\phi - \int_{\pi/4}^{3\pi/4} \cos^2\phi\sin\phi\, d\phi\right)$$

$$= 4\pi\sqrt{2}\left(-\cos\phi + \frac{1}{3}\cos^3\phi\right)\Big|_{\pi/4}^{3\pi/4} = 4\pi\sqrt{2}\left(\frac{2}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right)$$
$$= \frac{20\pi}{3}$$

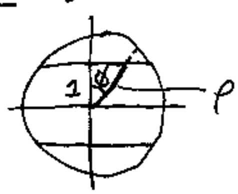
(c) Use the divergence theorem! $\operatorname{div} G = 1$,

so $\iint_D \vec{G} \cdot dS$, where $D$ is the entire closed surface of the drum

$$= \iiint_E 1 \cdot dV = \text{volume of drum}$$



a bit ugly, but in spherical coordinates;
easier to find volumes of top, bottom caps and subtract from volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(2\sqrt{2})$




$\cos\phi = \frac{1}{\rho}$, $\rho = \frac{1}{\cos\phi}$ inside limit, up to $\rho = \sqrt{2}$

so volume of top cap $= \int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\cos\phi}}^{\sqrt{2}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left(\frac{2\sqrt{2}}{3} - \frac{1}{3}\frac{1}{\cos^3\phi}\right)\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left(-\frac{2\sqrt{2}}{3}\cos\phi - \frac{1}{6\cos^2\phi}\right)\Big|_0^{\pi/4} d\theta = \int_0^{2\pi}\left(\left(-\frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{3}\right) - \left(-\frac{2\sqrt{2}}{3} - \frac{1}{6}\right)\right) d\theta = \frac{4\pi\sqrt{2}}{3} - \frac{5}{3}\pi$$

(9) (c) continued so volume of drum

$$= \frac{4}{3}\pi(2\sqrt{2}) - 2\left(\frac{4\pi\sqrt{2}}{3} - \frac{5}{3}\pi\right) = \frac{10\pi}{3} = \text{answer to (c)}$$

(d) $\vec{H} = \vec{F} + \vec{G}$, so $\iint_D \vec{H}\cdot dS = \iint_D (\vec{F}+\vec{G})\cdot dS$

$$= \iint_D \vec{F}\cdot dS + \iint_D \vec{G}\cdot dS$$

(where $D$ is the surface of the drum)

since $\vec{F} = \langle x, y, 0\rangle$ has no flux through the top or bottom of the drum (as the $z$ component is zero), then $\iint_D \vec{F}\cdot dS$ = answer to part (b) $= \frac{20\pi}{3}$

and $\iint_D \vec{G}\cdot dS = \frac{10\pi}{3}$ from part (c) so $\iint_D \vec{H}\cdot dS =$

$$= \frac{20\pi}{3} + \frac{10\pi}{3} = 10\pi$$

(e) hmm... we already found this for part (c) $= \frac{10\pi}{3}$.

Note $\operatorname{div}\vec{H} = 3$, so answer to part (d)

$$= \iiint_E \operatorname{div}\vec{H}\, dV = 3\iiint_E dV = 3\times\left(\frac{10\pi}{3}\right) = 10\pi$$

as we found in (d) because of the divergence Theorem.

(10) intersection $z = x^2+y^2+2x+2y-9$ and $z = 2x+2y$

so $x^2+y^2+2x+2y-9 = 2x+2y$, so $x^2+y^2-9=0$

intersection is just the circle $x^2+y^2=9$, with varying $z = 2x+2y$ (so its circular when looking down $z$)

Stokes' Thm: $\iint_S (\text{curl}\,\vec{F})\, dS = \oint_C \vec{F}\cdot d\vec{r}$

parametrize surface as $x^2+y^2\le 9$, $z = 2x+2y$

so $\vec{r}(x,y) = \langle x, y, 2x+2y\rangle$ with $x^2+y^2\le 9$ ...

and $\vec{r}_x\times\vec{r}_y = \langle -2,-2,1\rangle$

(10) continued now $\text{curl}\,\vec{F} = (z-3x)\vec{i} + (0-0)\vec{j} + (3z+2)\vec{k}$

so Stokes' Thm yields $\iint_R (\text{curl}\,\vec{F}) \cdot \langle -2,-2,1 \rangle \, dx\,dy$ where $R$ is $x^2+y^2 \le 9$

$$= \iint_R (-2z+6x+3z+2)\,dx\,dy$$

where $z = 2x+2y$ so $= \iint_R (8x+2y+2)\,dx\,dy$

convert to polar: $\int_0^{2\pi}\int_0^3 (8r\cos\theta + 2r\sin\theta + 2)\,r\,dr\,d\theta = 18\pi$

(11) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,F\,dV = \iiint_E (3x^2+3y^2+y)\,dV$

convert everything to cylindrical:

$$= \int_0^{2\pi}\int_0^3\int_{r^2+2r(\cos\theta+\sin\theta)-9}^{2r(\cos\theta+\sin\theta)} (3r^2 + r\sin\theta)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \big(6r^4\cos\theta + 6r^4\sin\theta + 2r^3\sin\theta\cos\theta + 2r^3\sin\theta - 3r^5 - 6r^4\cos\theta$$
$$- 6r^4\sin\theta + 27r^3 - r^4\sin\theta - 2r^3\sin\theta\cos\theta - 2r^3\sin\theta + 9r^2\sin\theta\big)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (-3r^5 + 27r^3 - r^4\sin\theta + 9r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{r^6}{2} + \frac{27r^4}{4} - \frac{r^5\sin\theta}{5} + 3r^3\sin\theta\right]\Big|_0^3 \,d\theta$$

$$= \int_0^{2\pi} \left(-\frac{729}{2} + \frac{2187}{4} - \frac{243\sin\theta}{5} + 81\sin\theta\right)d\theta$$

$$= \left[\frac{729\theta}{4} + \frac{162}{5}\cos\theta\right]_0^{2\pi} = \frac{729\pi}{2}$$