Name: Answer Key

Math 21a Midterm 2 Tuesday, December 7th 2004

Please circle your section:

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Eaman Eftekhary Walker Brumskine Tu/Th 10–11:30	Jeng-Daw Yu Serena Rezny Tu/Th 10–11:30	Sug Woo Shin Emily Morgan T/Th 11:30–1	

Question	Points	Score
1	8	
2	12	
3	14	
4	10	
5	16	
6	16	
7	12	
8	12	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order – move on to another problem if you find that you're stuck and are spending too much time on one problem.

To receive full credit on a problem, you need to justify your answers carefully (unless the question specifically says otherwise). Unsubstantiated answers will receive little or no credit! Please be sure to write neatly – illegible answers will also receive little or no credit.

If you need more space for a problem then use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

Please note that you are not allowed to use any notes or calculators during this test.

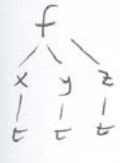
Good luck! Focus and do well!

Question 1. (8 points total)

Suppose the temperature in degrees Fahrenheit at the point (x, y, z) in a room is given by the function $f(x,y,z) = ze^{xy} + 50$. A spider and a fly are moving around in the room. The fly is zooming around so that its position is given by the vector function $\mathbf{r}(t) = \langle \cos(t),\ 2,\ t+1 \rangle$ at time t seconds. The spider is moving around on a web whose surface can be parametrized by $\mathbf{r}(u,v) = \langle 3u,\ u-v,\ 2v^2 \rangle$.

(a) (4 points) What is the rate of change of the temperature per second as experienced by the fly at time $t = 0$?

calculate $\frac{df}{dt}$ when $t=0$, so $\vec{r}(0) = \langle \cos(0), 2, 0+1 \rangle = \langle 1, 2, 1 \rangle$



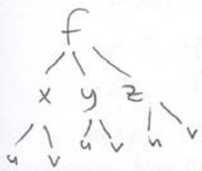
so $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

$= (zye^{xy})(-\sin t) + (zxe^{xy}) \cdot 0 + (e^{xy}) \cdot 1$

Now at time $t=0$, $x=1$, $y=2$, $z=1$

so $\left.\frac{df}{dt}\right|_{t=0} = 0 + 0 + e^{1 \cdot 2} \cdot 1 = e^2$

(b) (4 points) If the spider is currently at the point (3,0,2) moving along a v constant grid curve then calculate $\frac{\partial f}{\partial u}$ at its current location.



so $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u}$

$= (zye^{xy}) \cdot 3 + (zxe^{xy}) \, 1 + (e^{xy}) \cdot 0$

when $x=3$, $y=0$, $z=2$, get

$\frac{\partial f}{\partial u} = 0 + 6e^0 + 0 = 6$

Question 2. (12 points total)

You are wandering around in a strange Math 21a desert where the temperature in Celsius at the point (x, y) is given by the function $T(x,y) = e^{y-x^2}$.

(a) (4 points) You've stopped at the point (2,5). Suddenly you're feeling chilled and want to warm up. In what direction should you go to warm up as rapidly as possible? (give a unit direction vector for your answer)

Find ∇T at $(2,5)$:

$\nabla T = \langle -2x e^{y-x^2}, e^{y-x^2} \rangle$

at $(2,5)$, $\nabla T(2,5) = \langle -4e, e \rangle = e\langle -4, 1 \rangle$

unit vector in same direction: $\langle -\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle$

(b) (4 points) You're still at the point (2,5), but now you've decided that in fact it's pretty comfortable and you like the temperature after all. In which directions from that point is the temperature change equal to 0? (give unit direction vectors in your answer)

so set $D_{\vec{u}} T = 0$ and find $\vec{u}$

$D_{\vec{u}} T(2,5) = \nabla T(2,5) \cdot \vec{u} = \langle -4e, e \rangle \cdot \vec{u}$

First find vector with dot product $= 0$, i.e.

$\langle 1, 4 \rangle$ (just check $\langle -4e, e \rangle \cdot \langle 1, 4 \rangle = 0$ – cool)

solve for this by taking $\langle -4e, e \rangle \cdot \langle a, b \rangle = 0$, so $(-4e)a + eb = 0$ $\Rightarrow b = 4a$

also $\langle -1, -4 \rangle$ (opp. direction) works too

so (unit) direction vectors are $\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$ and $\langle \frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}} \rangle$

(c) (4 points) Now figure out the coordinates of all the points where there is no increase or decrease in temperature in the $\langle 1,1 \rangle$ direction.

unit vector for $\langle 1, 1 \rangle$ direction is $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

so set $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} T = 0$, or $\nabla T \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0$

$= \langle -2x e^{y-x^2}, e^{y-x^2} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0$

so just $\frac{1}{\sqrt{2}}(-2x+1)e^{y-x^2} = 0 \Rightarrow -2x + 1 = 0$, $x = \frac{1}{2}$

y can be anything

so points are all points with coordinates $(\frac{1}{2}, c)$ $c \in \mathbb{R}$

Question 3. (14 points total)

Engineer Dolbert is put in charge of a new project with a budget constraint of \$18,000. The success of the project depends on how well he distributes the workload between the engineering department (that costs \$40 per hour) and the outside management consulting firm McKonzey (that charges \$120 per hour).

The financial success of the project $S(x, y)$ in dollars is given by $S(x, y) = xy^2$ where x is the number of hours put into the project by the engineering department, and y is the number of hours put in by McKonzey.

(a) (8 points) Use the method of Lagrange multipliers to figure out how Dolbert should distribute the workload to achieve maximal success for the project.

Find constraint → budget $= 18000$,

if $x =$ hours engineering, $y =$ hours McKonzey,

then $40x + 120y = 18000$ i.e. $g(x,y) = 40x + 120y = 18{,}000$

and $S(x,y) = xy^2$

So Lagrange: $\nabla g = \lambda \nabla S$

$$\langle 40, 120\rangle = \lambda \langle y^2, 2xy\rangle$$

① $40 = \lambda y^2$ ③ $40x + 120y = 18000$

② $120 = \lambda 2xy$

multiply ① by 3: $120 = 3\lambda y^2 = \lambda 2xy$ by ②

so $3\lambda y^2 - \lambda 2xy = 0$, or $\lambda y(3y - 2x) = 0$

So either $\lambda = 0$, not possible by ①

or $y = 0$, not possible by ①

or $3y - 2x = 0 \Rightarrow x = \frac{3}{2}y$.

Then by ③ $40(\frac{3}{2}y) + 120y = 18000$

so $60y + 120y = 180y = 18000$, $y = 100$

so $x = \frac{3}{2}y = 150$

so... 100 hours of McKonzey,

150 hours to engineering dept.

(clearly a max → check x or $y = 0 \Rightarrow S(0,y) = S(x,0) = 0 < 150 \cdot (100)^2$

Question 3 continued

(b) (6 points) After completing the original project, Dolbert's boss now wants to do a small follow-up project where this time he wants to see a financial success of \$1,500. What is the minimal budget that Dolbert needs to achieve the goal of \$1,500 by running another project (again solving this problem using Lagrange multipliers and using the same costs for the engineering department and McKonzey and the same success formula $S(x, y) = xy^2$ as before)?

Same equations as before, but flipped:

now $S(x,y) = xy^2 = 1500$

$g(x,y) = 40x + 120y$ needs to be minimized

again ① $40 = \lambda y^2$

② $120 = \lambda 2xy$

now ③ $xy^2 = 1500$

so again from ①,② $x = \frac{3}{2}y$,

so from ③ $\left(\frac{3}{2}y\right)y^2 = \frac{3}{2}y^3 = 1500$

$y^3 = \frac{2}{3}1500 = 1000$

$y = 10$

$\Rightarrow x = 15$

budget $= g(15,10) = 40 \cdot 15 + 120 \cdot 10$

$= 600 + 1200 = \$1500$

clearly min, as for example if

$x = 1500$ $y = 1$ (to get $S(1500, 1) = 1500$)

budget $= 40 \cdot 1500 + 120 \cdot 1 = \$60,120$

$> \$1800$

Question 4. (10 points total)

(a) (4 points) Which of the following functions are solutions of the wave equation $\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial x^2}$?
Write down "solution" next to each of the functions that is a solution.

Solution (i) $3x^2 + 12t^2 + 1999$ X (ii) $\cos(4t)\cos(4x)$

Solution (iii) $1999\cos(2t)\sin(x)$ Solution (iv) $x^3 + 12xt^2$

for instance for (i) $\frac{\partial^2 f}{\partial t^2} = 24 = 4\left(\frac{\partial^2 f}{\partial x^2}\right) = 4 \cdot 6$

and for (ii) $\frac{\partial^2 f}{\partial t^2} = -16\cos(4t)\cos(4x) \neq 4\left(\frac{\partial^2 f}{\partial x^2}\right) = 4\left(-16\cos(4t)\cos(4x)\right)$

(b) (3 points) Now find a function $f(x,t)$ that is a solution to the same wave equation $\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial x^2}$ satisfying the initial condition $f(0,0) = 2004$ (note, you don't need a general solution to do this – think about the solution functions that you found in part (a)!)

Hmm... scanning through solutions (i) (iii) and (iv) from (a) clearly there's nothing special about the constant 1999 in (i), and there $f(0,0) = 1999$, so how about

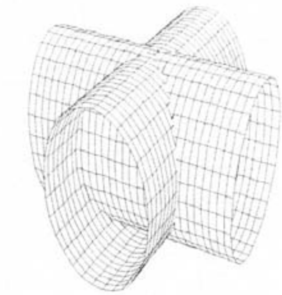
$3x^2 + 12t^2 + 2004 \rightarrow$ solution w/ $f(0,0) = 2004$

(c) (3 points) Now find a different function $f(x,t)$ that is a solution to the same wave equation $\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial x^2}$ satisfying the initial conditions $f(0,0) = 0$ and $f_x(0,0) = 2004$

Could work with solutions (i) or (iv), but easier to go with (iii) as if $f(x,t) = 1999\cos(2t)\sin(x)$ then $f(0,0) = 0$, and $f_x(x,t) = 1999\cos(2t)\cos(x)$ so $f_x(0,0) = 1999$. So try $2004\cos(2t)\sin(x)$

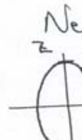
Question 5. (16 points total)

Consider the intersection of two elliptical cylinders whose equations, respectively, are $4x^2 + z^2 = 4$ and $4y^2 + z^2 = 4$, as shown on the left. Find the volume of the solid region that's common to the interiors of both cylinders, i.e. the region shown on the right.



There are a number of ways to set up an integral to compute this volume. Perhaps the most straightforward is to set up a triple integral $\iiint_E dV$ and integrate in either the x or y direction first, for instance if we do x first then note the limits of x come from $4x^2+z^2=4$, or $4x^2 = 4-z^2$, $x = \pm \frac{\sqrt{4-z^2}}{2}$, so $-\frac{\sqrt{4-z^2}}{2} \leq x \leq +\frac{\sqrt{4-z^2}}{2}$

Next we're left with the elliptical region in the yz plane



and again from $4y^2+z^2$ we have limits for y: $-\frac{\sqrt{4-z^2}}{2} \leq y \leq \frac{\sqrt{4-z^2}}{2}$. Finally $-2 \leq z \leq 2$

So... $\int_{-2}^{2} \int_{-\frac{\sqrt{4-z^2}}{2}}^{\frac{\sqrt{4-z^2}}{2}} \int_{-\frac{\sqrt{4-z^2}}{2}}^{\frac{\sqrt{4-z^2}}{2}} dx\,dy\,dz = \int_{-2}^{2} \int_{-\frac{\sqrt{4-z^2}}{2}}^{\frac{\sqrt{4-z^2}}{2}} 2\left(\frac{\sqrt{4-z^2}}{2}\right) dy\,dz$

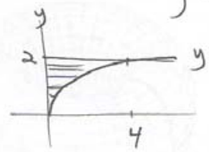
$= \int_{-2}^{2} 2\left(\frac{\sqrt{4-z^2}}{2}\right) 2\left(\frac{\sqrt{4-z^2}}{2}\right) dz = \int_{-2}^{2} (4-z^2)\,dz = 4z - \frac{1}{3}z^3 \Big|_{-2}^{2} = \frac{32}{3}$

Question 6. (16 points total)

(a) (8 points) Evaluate the double integral $\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\,dy\,dx$.

Can't do $\int e^{y^3}\,dy$, so try reversing order...

region:



$y = \sqrt{x}$ or $y^2 = x$

x first $0 \le x \le y^2$

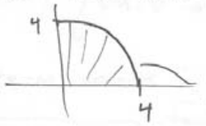
$$\ldots \int_0^2 \int_0^{y^2} e^{y^3}\,dx\,dy = \int_0^2 y^2 e^{y^3}\,dy$$

$$= \frac{1}{3} e^{y^3} \Big|_0^2 = \frac{1}{3}\left(e^8 - 1\right)$$

(b) (8 points) Evaluate the double integral $\int_0^4 \int_0^{\sqrt{16-x^2}} e^{(x^2+y^2)}\,dy\,dx$.

Yuck again ... here x^2+y^2 and the suspicious $\sqrt{16-x^2}$ limit should ring bells about polar coordinates (reversing order does nothing here)

So ... region:



aha! circles! (or <u>part</u> of one)

$y = \sqrt{16-x^2}$

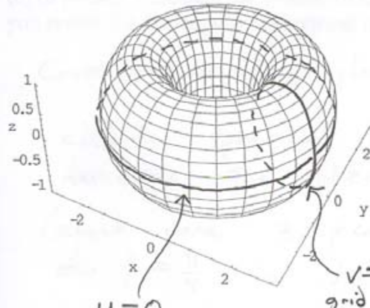
or $x^2+y^2 = 16 = 4^2$

So ... convert limits $0 \le r \le 4$, $0 \le \theta \le \frac{\pi}{2}$, integrand $e^{(x^2+y^2)} = e^{r^2}$, $dy\,dx \ldots r\,dr\,d\theta$,

get $\int_0^{\pi/2} \int_0^4 r e^{r^2}\,dr\,d\theta = \int_0^{\pi/2} \left(\frac{1}{2} e^{r^2} \Big|_{r=0}^4 \right) d\theta = \frac{\pi}{4}\left(e^{16} - 1\right)$

Question 7. (12 points total)

Find the surface area of the doughnut pictured below that is parametrized by the vector function
$\mathbf{r}(u,v) = \langle (2+\cos u)\cos v,\ (2+\cos u)\sin v,\ \sin u \rangle$



Okay, know S.A. formula

$$= \iint_{u,v \text{ limits}} |\vec{r}_u \times \vec{r}_v| \, du\,dv \ldots$$

u = 0
grid curve
("big doughnut" circle)

v = 0
grid curve
(little circle)

note to get whole doughnut need to let $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ (to get the little circles and the big doughnut circle itself)

$$\vec{r}_u \times \vec{r}_v = \begin{array}{c} \\ \vec{r}_u \\ \vec{r}_v \end{array} \begin{vmatrix} i & j & k \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -(2+\cos u)\sin v & (2+\cos u)\cos v & 0 \end{vmatrix}$$

$$= \left\langle -(2+\cos u)\cos v \cos u,\ \pm (2+\cos u)\sin v \cos u,\ -(2+\cos u)\sin u \overbrace{(\cos^2 v + \sin^2 v)}^{1} \right\rangle$$

doesn't matter as we're about to square everything!

$$\text{so } |\vec{r}_u \times \vec{r}_v| = \sqrt{(2+\cos u)^2 \Big[\overbrace{\underbrace{\cos^2 v \cos^2 u + \sin^2 v \cos^2 u}_{\cos^2 u} + \sin^2 u}^{=1} \Big]}$$

same $(2+\cos u)$ term occurs in all 3 components of $\vec{r}_u \times \vec{r}_v$, so factor out first to keep algebra simple!

$$\text{so } = \sqrt{(2+\cos u)^2} = 2 + \cos u$$

$$\text{so S.A.} = \int_0^{2\pi}\int_0^{2\pi} (2+\cos u)\, du\, dv$$

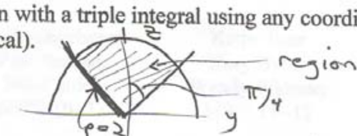
$$= \int_0^{2\pi}\int_0^{2\pi} 2\, du\,dv + \underbrace{\int_0^{2\pi}\int_0^{2\pi} \cos u\, du\, dv}_{\to 0} = 2\pi \cdot 2\pi \cdot 2 = 8\pi^2$$

Question 8. (12 points total)

Consider the region in space that is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$.

(a) (8 points) Calculate the volume of this region with a triple integral using any coordinate system you prefer (i.e. rectangular, cylindrical or spherical).

Cross section in yz plane:



clearly spherical looks good as simply $\rho = 2$ describes the sphere and $\phi = \frac{\pi}{4}$ describes the cone

(check cone: $z = \rho\cos\phi = \sqrt{x^2+y^2} = r = \rho\sin\phi$, so $\cos\phi = \sin\phi$ $\Rightarrow \phi = \frac{\pi}{4}$ (or just check it's a $\frac{\pi}{4} = 45°$ angle cone (side))

so integral for volume is just

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/4}\left(\frac{1}{3}\rho^3\Big|_{\rho=0}^{2}\right)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\frac{8}{3}\left(-\cos\phi\right)\Big|_{\phi=0}^{\pi/4} d\theta = 2\pi\cdot\frac{8}{3}\left(-\frac{1}{\sqrt{2}}-(-1)\right) = \frac{16\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

(b) (4 points) Now write down a triple integral that gives the volume of the same region, but using a different coordinate system from the one you used in part (a). (just write it down - you don't need to evaluate it (again)!)

Okay... cylindrical next easiest as cone is just $z = r$, and sphere $x^2+y^2+z^2=4$ is $r^2+z^2=4$ or $z=\sqrt{4-r^2}$,

so... intersect when $z=r$ and $r^2+z^2=4$, or $r^2+r^2=4 \Rightarrow r=\sqrt{2}$

Cylindrical: $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

Rectangular: same idea $r=\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}$ while x, y vary over circular region $x^2+y^2 \le (\sqrt{2})^2$ ... $(y=\sqrt{2-x^2})$

$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$$